

ORIGINAL RESEARCH ARTICLE

On some Asymptotic Properties of the Extended Cosine Burr XII Distribution

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ABSTRACT

In statistics, asymptotic functions provide a powerful framework for understanding the behavior of functions or statistical procedures within the limits of certain parameters. Also, in reliability, mean residual life assumes a critical role in evaluating the durability and effectiveness of diverse components. In addition, order statistics are essential tools in extreme value analysis. In this article, a new probability model is proposed based on the extended cosine-G family of distributions, called extended cosine Burr XII (ECSBXII) distribution. Some important basic properties are derived. We derive and study the asymptotic of mean residual life and asymptotic of the extreme order statistics of the ECSBXII distribution. An illustrative application of asymptotic MRL of ECBXII is given using simulation studies.

INTRODUCTION

In studies related to reliability and other statistical analyses, the concept of mean residual life (MRL) holds substantial importance, featuring a broad spectrum of applications, refer to (Chen et al., 2023). MRL assumes a critical role in evaluating the durability and effectiveness of diverse components, spanning from mechanical systems to software and extending to survival analysis in medical research. Its capability to predict the potential failure of a system underscores its significance. This discussion explores the importance and practical uses of mean residual life, highlighting how this concept empowers reliability engineers and statisticians to improve the quality, safety, and efficiency of various systems and processes. A recent study by (Hall & Wellner 2020) extensively delves into the estimation of mean residual life, contributing valuable insights to this field.

Definition 1. *The mean residual life (MRL) is the average time above t until failure, provided a component or subject has survived up to the time t . Given a random variable X , the MRL is expressed by*

$$M_X(t) = E(X - t | X > t),$$

$$M_X(t) = \int_0^{\infty} \frac{R(x+t)}{R(t)} dx, \quad (1)$$

Where $R(t)$ is the survival function X .

On the other hand, order statistics play a substantial role in theoretical studies and practice, particularly in dealing with the minimum, maximum, and range and in studying a particular X_i . Order statistics are crucial tools in extreme value study, analysis of random phenomena in quality control, and life testing, among other fields. One can see (Chua et al., 2010), (Abba et al., 2023), (Serinaldi et al., 2020), and (Koutras & Koutras 2020).

Definition 2. *Given an n -size random sample denoted by X_1, X_2, \dots, X_n , $i = 1, 2, \dots, n$, the order statistics are the $X_{(1)}, X_{(2)}, \dots, X_{(n)}$, expressed in function of X_1, X_2, \dots, X_n , such that $X_{(1)} = \inf(X_1, X_2, \dots, X_n)$, $X_{(n)} = \sup(X_1, X_2, \dots, X_n)$ and $P(X_{(1)} \leq X_{(2)} \leq \dots, X_{(n)}) = 1$.*

In this work, we derive a new probability model from so-called extended cosine Burr XII (ECS- BXII) distribution, and some basic properties are derived. We discussed the asymptotic of MRL and asymptotic of the extreme order statistics of the ECSBXII distribution by initially obtaining some asymptotic functions of the ECSBXII such as the asymptotic of the cumulative distribution function and asymptotic of the reliability function.

The rest of the work follows: in Section 2, we derive the ECSBXII. In Section 3, the asymptotic of MRL and asymptotic of the extreme order statistics of the ECS- BXII are discussed. Conclusion in section 4.

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THE ECSBXII DISTRIBUTION

(Chen et al., 2023) Proposed the general form of the CDF of the extended cosine-G (ECSG) family of distribution as

$$D(x) = 1 - \left(1 - \cos\left(\frac{\pi}{2}s(x, \xi)\right)\right)^\alpha \tag{2}$$

The related probability density function (PDF) reliability function are respectively given as

$$d(x) = \alpha \frac{\pi}{2} g(x, \xi) \left(1 - \cos\left(\frac{\pi}{2}s(x, \xi)\right)\right)^{\alpha-1}, \tag{3}$$

$$R(x) = \left(1 - \cos\left(\frac{\pi}{2}s(x, \xi)\right)\right)^\alpha, \tag{4}$$

Where $g(x, \xi)$ and $s(x, \xi)$ are the PDF and reliability function of any valid distribution, ξ a vector parameters. In our case, we consider $g(x, \xi)$ and $s(x, \xi)$ to be from the BurrXII distribution proposed by (Al-Essa et al., 2023) with density functions and reliability functions:

$$f(x) = \frac{\alpha\lambda\theta\pi}{2} x^{\theta-1}(1+x^\theta)^{-\lambda-1} \sin \sin\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right) \left(1 - \cos \cos\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right)\right)^{\alpha-1} \tag{7}$$

$$R(x) = 1 - \left(1 - \cos \cos\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right)\right)^\alpha \tag{8}$$

The failure rate (FR) function of the ECBXII is derived as

$$h(x) = \alpha\lambda\pi x^{\theta-1}(1+x^\theta)^{-\lambda-1} \frac{\sin \sin\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right) \left(1 - \cos \cos\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right)\right)^{\alpha-1}}{2 \left(1 - \left(1 - \cos \cos\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right)\right)^\alpha\right)} \tag{9}$$

Figure 1 shows the plot of the possible shape of the ECSBXII density functions and FR function for some parameter values.

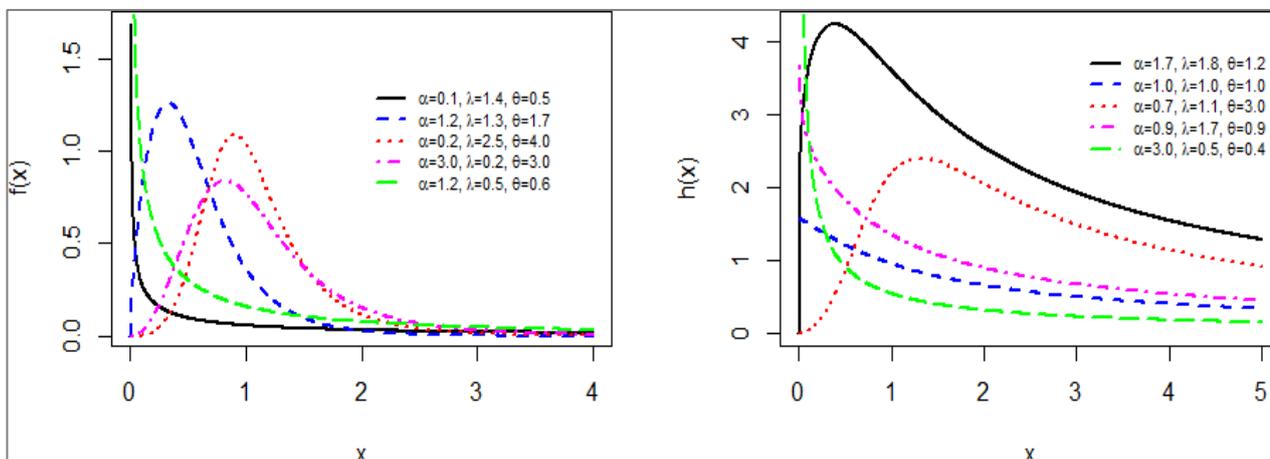


Figure 1: Plot of the PDF and FR of the ECSBXII for some parameter values

$$g(x, \lambda, \theta) = \lambda\theta x^{\theta-1}(1+x^\theta)^{-\lambda-1},$$

$$s(x, \lambda, \theta) = (1+x^\theta)^{-\lambda-1}, \tag{5}$$

respectively, where $x, \lambda, \theta > 0$. There are several extensions and applications of BXII in the literature; for example, exponentiated exponential Burr XII was proposed by (Badr & Ijaz, 2021), complementary exponentiated Burr XII Poisson (Muhammad, 2017), and generalized of the Burr XII-Poisson (Muhammad, 2016), among others. For some recent applications of BXII one can see (Du & Gui, 2022). For statistical inference of burr-xii distribution with adaptive type II progressive censored strategies with competing risks, and (Saini & Garg, 2022) for the reliability inference of multicomponent stress-strength model by employing the Burr XII distribution under progressively first-failure censored samples, etc.

Hence, the CDF of the ECSBXII is derived using (5) and the corresponding PDF and reliability functions as:

Hence, the CDF, PDF and reliability functions of the ECSBXII are derived as follows:

$$F(x) = \left(1 - \cos \cos\left(\frac{\pi}{2}(1+x^\theta)^{-\lambda}\right)\right)^\alpha \tag{6}$$

THE ASYMPTOTIC OF MRL AND ASYMPTOTIC OF THE EXTREME ORDER STATISTICS FOR THE ECSBXII

Asymptotic functions play a crucial role in both mathematics and statistics. For example, in statistics, asymptotic theory is employed to make inferences about population parameters based on sample data. Asymptotic results often provide insights into the behavior of estimators and test statistics as the sample size becomes large. In both mathematics and statistics, asymptotic functions provide a powerful framework for understanding the behavior of functions, algorithms, and statistical procedures in the limit as certain parameters, such as input size or sample size, approach infinity. In this section, we derive and discuss the asymptotic of mean residual life and asymptotic of the extreme order statistics for the ECSBXII. First, we provided some important Lemma as follows:

Lemma 3.1. (Muhammad et al., 2021). *The asymptotic of the CDF in (2) and reliability function in (4) are given by.*

$$F(x) \sim \alpha \cos \cos \left(\frac{\pi}{2} s(x, \xi) \right). \tag{10}$$

$$R(x) \sim \left(\frac{1}{2} \right)^\alpha \text{Sin}^{2\alpha} \left(\frac{\pi}{2} s(x, \xi) \right) \sim \left(\frac{1}{2} \right)^\alpha \left(\frac{\pi}{2} \right)^{2\alpha} s^{2\alpha}(x, \xi) \tag{11}$$

The asymptotic of MRL for the ECSBXII

The MRL of the ECSBXII distribution is no doubt complicated to derive because it contains expressions that require huge approximations and expansions, one can see (Muhammad et al., 2021), (Muhammad et al., 2021), (Muhammad et al., 2023) to verify. In some models, the MRL is not compatible by taking direct integral at any given parameter; Monte Carlo integral is needed to be employed to evaluate MRL using the sampling technique, one can see (Ahsanullah et al., 2013), (Arnold et al., 1992), Abba & Wang (2023). Related studies regarding the derivation of asymptotic MRL of some models can be found in some chapters of Muhammad (2023) & Muhammad & Liu (2021), among others.

Now, we utilized Lemma 3.1 and the given Lemma below to derive the asymptotic of MRL for the ECSBXII distribution:

Lemma 3.2. (Gradshteyn & Ryzhik, 2007, p. 315) *u, t, μ > 0 be real numbers such that μ > μ > 0, then*

$$\int_t^\infty y^{-u} (y - t)^{\mu-1} dy = t^{\mu-u} B(u - \mu, \mu),$$

Where $B(\dots)$ is a beta function.

Theorem 3.3. *Let X be a random with ECSBXII, for $\alpha\theta\lambda > \frac{1}{2}$, and for a very large $t > 0$, i.e., as $t \rightarrow \infty$ the mean residual life is*

$$M_X(t) \sim tB(2\alpha\theta\lambda - 1, 1).$$

Proof. From the (11), the asymptotic of the reliability function of the ECSBXII as $x \rightarrow \infty$ is

$$R(x) \sim \left(\frac{1}{2} \right)^\alpha \left(\frac{\pi}{2} \right)^{2\alpha} (1 + x^\theta)^{-2\alpha\lambda} \sim \left(\frac{1}{2} \right)^\alpha \left(\frac{\pi}{2} \right)^{2\alpha} x^{-2\alpha\theta\lambda} \tag{12}$$

Thus, we can derive the asymptotic of the $M_X(t)$ for the ECSBXII as $t \rightarrow \infty$. By letting $u = x + t$;

$$M_X(t) = \int_0^\infty \frac{R(x+1)}{R(t) + t} dx \sim t^{2\alpha\theta\lambda} \int_0^\infty (x + t)^{2\alpha\theta\lambda} dx$$

$$M_X(t) \sim t^{2\alpha\theta\lambda} \int_t^\infty u^{-2\alpha\theta\lambda} du$$

by applying the Lemma 3.2,

$$M_X(t) \sim tB(2\alpha\theta\lambda - 1, 1).$$

The asymptotic of the extreme order statistics for the ECSBXII

Let $X_1, X_2, \dots, X_n, n \geq 1$, be an ordered sample from a probability distribution, then the PDF of the j^{th} - order statistic represented by $f_{j:n}(x)$, is expressed as

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)(F(x))^{j-1} (1 - F(x))^{n-j},$$

Where, for $F(x)$ and $f(x)$ are given in are given in (6) and (7), respectively.

Now, we can derive the extreme order statistics asymptotic distributions, i.e., $X_{1:n}$ and $X_{n:n}$ from $X_1, X_2, X_3, \dots, X_n$, follow ECSBXII. Based on the detailed information in (Al-Essa et al., 2023, p. 50130–50144.), let $d \rightarrow$ denote convergence in distribution, let W denote a random variable with cdf, G , then, claiming that the cdf, F is contained in the domain of maximal attraction of G is equally as saying $\frac{(x_{n,n} - a_n)}{b_n} d \rightarrow W$, on condition that a sequence $\{a_n\}$ and $\{b_n > 0\}$ exist. Take W^* as a random variable with cdf, G^* , then, to claim that the cdf F is contained in the domain of minimal attraction of G^* is equally as saying $\frac{(x_{1,n} - a_n^*)}{b_n^*} d \rightarrow W^*$, so long as the sequence $\{a_n^*\}$ and $\{b_n^* > 0\}$ exist.

Theorem 3.4. Let $\{X_i\}_{i=1}^n$ be a random sample from ECSBXII, let $B_n = \frac{(X_{n:n} - a_n)}{b_n}$ then $B_n \xrightarrow{d} B$ indicated that

$$P(B_n \leq x) = G(x) = e^{-x^{-2\alpha\theta\lambda}},$$

for every point $x \in R$ of $G(x)$ for which $G(x)$ is continuous. The normalizing constant can be determined following Theorem 8.3.4 of (Al-Essa et al., 2023, p. 50130–50144.) as $a_n = 0$ and $b_n = F^{-1}\left(1 - \frac{1}{n}\right)$.

Proof. From the Theorem 8.3.2 of (Al-Essa et al., 2023, p. 50130–50144.) [4], and the asymptotic of $R(x)$ in equation (12) above, we have

$$\frac{R(tx)}{R(t)} \sim \frac{(tx)^{-2\alpha\theta\lambda}}{t^{-2\alpha\theta\lambda}} = x^{-2\alpha\theta\lambda}.$$

SIMULATION

Next, we fitted the asymptotic of MRL of ECSW to some suitable simulated data set for demonstration. We used simulation due to the necessary condition on the parameters for the asymptotic of MRL of ECSW to exist, because finding suitable real data is not certain.

We simulated data in Table 2, 3, 4 and 5 from ECSW as follows:

Table 1: Data generating process

1. Choose the sample size n and the values of parameters α, θ and λ
2. Generate random sample $U_i \sim \text{uniform}(0,1)$ distribution, $i = 1, 2, 3, \dots, n$.
3. Generate random sample $X_i \sim \text{ECSBXII}$ $i = 1, 2, 3, \dots, n$. Using
$X_i = \left[\left(\frac{\pi}{2} \text{across} \left(1 - (1 - u_i)^{\frac{1}{\alpha}} \right) \right)^{\frac{-1}{\lambda}} - 1 \right]^{\frac{1}{\theta}}$

Table 2: First simulated data set: We have drawn this data of size 50 by choosing $\alpha=0.08, \theta=2.5$ and $\lambda=0.5$, this satisfied $2\alpha\theta\lambda > 1$.

1.0526863, 0.7592534, 0.6199662, 0.8458926, 1.4466212, 0.9477421, 3.0628151, 1.1563635, 1.0027766, 4.3769630, 1.3499269, 2.0325724, 4.0858710, 12.2951947, 1.5724289, 1.3628819, 2.9104412, 0.3677692, 0.9983797, 0.5418595, 0.6193770, 0.4555360, 0.9372574, 0.6227970, 1.6177902, 0.7998012, 0.6682441, 1.3015635, 1.1718817, 0.9652878, 0.2432195, 2.5558146, 0.7921677, 0.6560692, 1.8018975, 2.1471022, 0.7028552, 3.2714242, 0.6163472, 1.3733793, 1.0069490, 0.9576615, 1.6597047, 0.8709967, 0.5631992, 1.7340274, 0.7442131, 0.9937960, 0.6482892, 5.134060
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Theorem 3.5. Let $\{X_i\}_{i=1}^n$ be a random sample from ECSBXII, let $B_n^* = \frac{(X_{1:n} - a_n^*)}{b_n^*}$ then $B_n^* \xrightarrow{d} B^*$ is implies that

$$P(B_n^* \leq x) = G^*(x; \theta) = 1 - e^{-x^\theta},$$

For every point $x \in R^+$ of $G^*(x; \theta)$ for which $G^*(x; \theta)$ is continuous. The normalizing constant can be derived the Theorem 8.3.6 of (Al-Essa et al., 2023, p. 50130–50144.) [4], thus, $a_n^* = 0$ and $F^{-1}\left(\frac{1}{n}\right)$.

Proof. From the Lemma 3.1 equation (11), and the Theorem 8.3.6 of (Al-Essa et al., 2023, p. 50130–50144.), we get,

$$\frac{F(F^{-1}(0) + tx)}{F(F^{-1}(0) + t)} \sim \frac{\cos \cos \left(\frac{\pi}{2} (1 + (tx)^\theta)^{-\lambda} \right)}{\cos \cos \left(\frac{\pi}{2} (1 + t^\theta)^{-\lambda} \right)} = x^\theta.$$

Table 3: Second simulated data set: We have drawn this data of size 50 by selecting $\alpha=0.8$, $\theta=0.9$ and $\lambda=1.9$, also satisfied $2\alpha\theta\lambda>1$.

0.1487469904,	0.3497698184,	0.3270217450,	0.1587141255,	0.0001295949,	0.3274437413,	0.0431751702,
0.0883832411,	0.1159976864,	0.1833147067,	0.0304904596,	0.6717685272,	0.0574652540,	0.0124909890,
0.5219354780,	0.7783300370,	0.4892950387,	0.0131813038,	0.0464725191,	0.7688078808,	0.1408983738,
0.1744707690,	0.0247276823,	0.3090570888,	0.4993275507,	0.2267023790,	1.0682109251,	0.0198553691,
0.0637601647,	0.4667411720,	0.0441704883,	0.5777042682,	0.0956217975,	0.1482966344,	0.0344888000,
0.5799731889,	0.0765468260,	0.4882239330,	2.5517335519,	0.0316012322,	0.2174329972,	0.3089546985,
0.2462696853,	0.3977333498,	0.1824579543,	0.4428262937,	0.0038487734,	0.8168070320,	0.1687887830,
0.7258260875						

Table 4: Third simulated data set: We simulated the data of size 50 by selecting $\alpha=1.0$, $\theta=1.0$ and $\lambda=1.0$, also satisfied $2\alpha\theta\lambda>1$.

0.317424722	0.436271523	0.283245708	0.211991164	0.412140211	0.747203967
0.415501799	0.395017732	0.824139527	0.404383786	0.369420448	1.450366517
0.007041701	0.628569242	0.145155315	1.029257875	0.285588171	0.031727335
0.299336555	0.108554273	0.611769540	1.724334625	1.548385695	0.281150165
0.180429692	0.997998484	0.290885684	0.760445347	0.414530449	2.694521631
1.072451123	1.913158077	0.099198306	0.447055353	2.289521445	0.403036248
0.197310956	0.659403472	0.430110769	0.647715953	1.882792993	0.748080428
0.176273293	0.126879159	0.224925179	0.293758877	0.063075250	6.380805968
0.334256376	0.618229284				

Table 5: Four simulated data set: We simulated the data of size 50 by selecting $\alpha = 0.9$, $\theta = 0.8$ and $\lambda = 1.6$, also satisfied $2\alpha\theta\lambda > 1$.

7.866669412	0.193544049	0.233349655	0.322319552	1.052919107	0.121931824
0.148202018	0.126403365	0.944103201	0.430205515	0.338933892	0.271518993
0.060368537	0.273225421	0.391951481	0.267072678	0.832361330	0.029514171
0.006185465	0.024960830	1.011453127	1.031239638	0.299797742	0.151558890
1.131417000	0.004871563	0.047579686	3.975409945	0.011196773	0.111160054
0.264921805	0.383905373	0.433469446	0.316959442	0.233411001	0.857271256
2.866480859	1.772993497	0.630951587	0.054277414	1.760295934	0.837123919
0.282658501	0.124953075	0.394427667	0.607135536	0.304025434	0.058698082
0.020184320	0.559280036				

Figures 2, 3, 4 and 5 are the plots of the MRL and the asymptotic of MRL of ECSW; it can be seen that the estimation was good and that the asymptotic of MRL of ECSW approximates the empirical MRL very well.

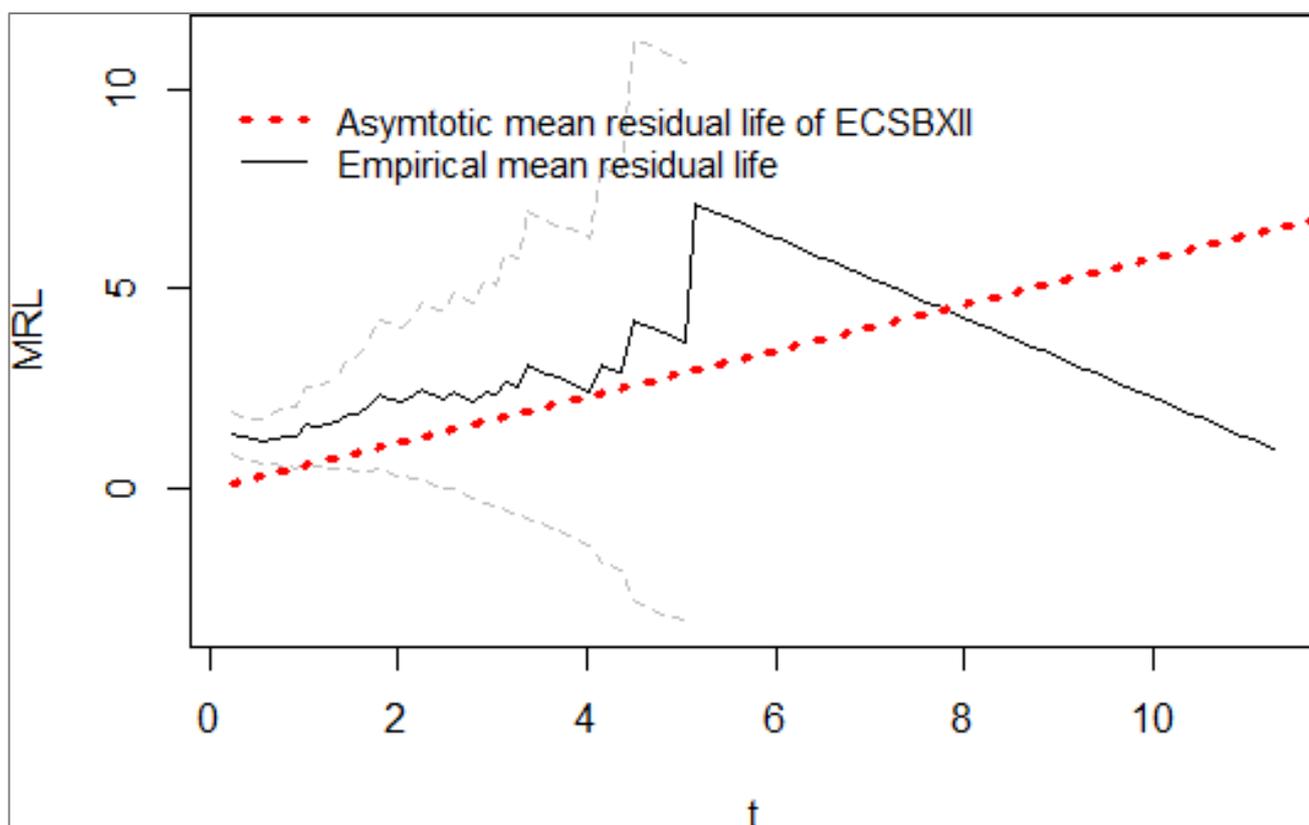


Figure 2: Plots of MRL and asymptotic of MRL of ECSW for the first simulated data

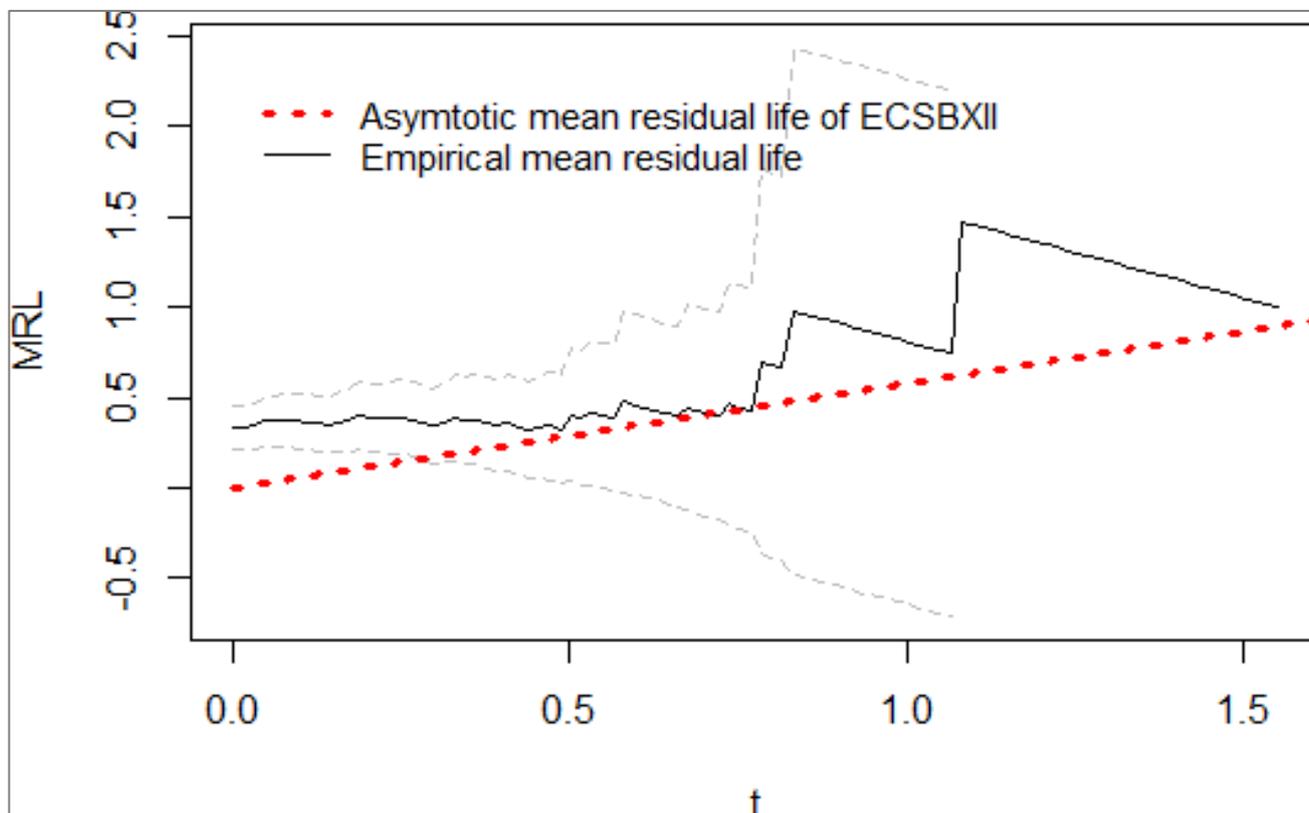


Figure 3: Plots of MRL and asymptotic of MRL of ECSW for the second simulated data

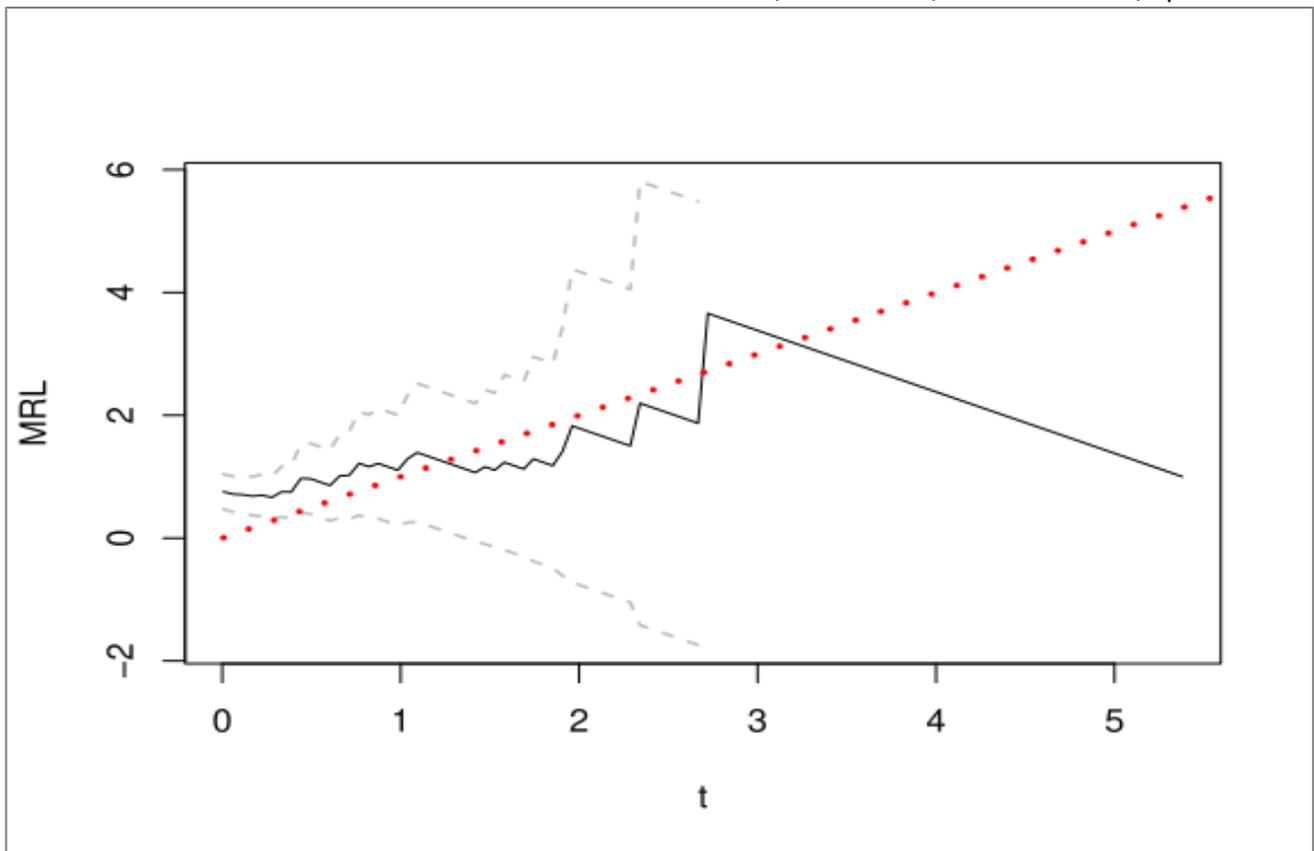


Figure 4: Plots of MRL and asymptotic of MRL of ECSW for the third simulated data

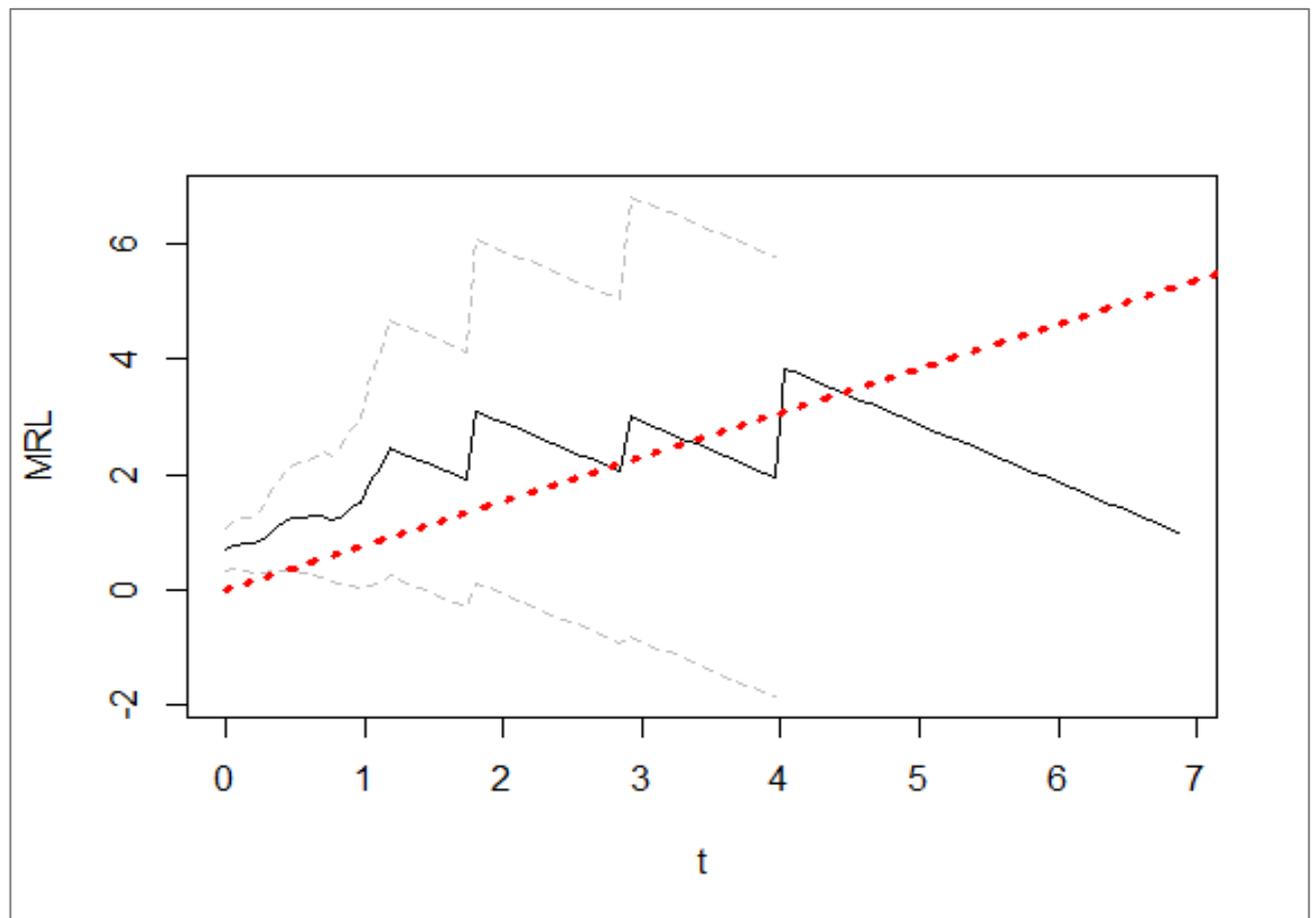


Figure 5: Plots of MRL and asymptotic of MRL of ECSW for the fourth simulated data

CONCLUSIONS

In this paper, a new probability model called extended cosine Burr XII distribution is proposed. Some important basic properties are derived. We derive and study the asymptotic of mean residual life and asymptotic of the extreme order statistics of the ECSBXII distribution. We demonstrated the asymptotic MRL of ECBXII performance using simulated data, and the result was satisfactory. These properties are important tools in statistical studies. Further studies can be conducted by applying these important tools in reliability analysis and extreme value analysis using the ECSBXII.

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