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ORIGINAL RESEARCH ARTICLE

A Linear Approximation of the Non-linear Modified Langumir and Van der Pol Differential Equations by the Application of the Generalized Sundman Transformation

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ABSTRACT

The non-linear ordinary differential equations of Langumir and Van der Pol are challenging to solve analytically. Thus, this work aims to convert these non-linear equations into linear form so that they may be easily solved. Assuming that the coefficients of the two equations meet the linearizability requirements, they are presented in the appropriate linearizable formats. After achieving this, the generalized Sundman transformation was used to linearize the equations. The formulae $u(t) = F(x, y),\ dt = G(x, y)dx,\ F_yG \neq 0$ defines the nonpoint transformation known as the generalized Sundman transformation (GST). Basic solutions for the two equations were obtained upon application of the GST. The conventional approach of variation of parameters was used to solve the linear equations that emerged from the linearization process.

INTRODUCTION

The Langumir equation shows how molecules are attached to a solid surface, especially in surface chemistry and adsorption. It illustrates how gas molecules remain fixed to a surface by having a set number of identical sites and no interactions between adsorbed molecules. In electrochemical systems, the Langumir equation can be applied to explain how ions or molecules are adsorbed onto electrode surfaces, enabling electrochemical reactions and sensor performance.

In the area of chemical engineering and physical chemistry, Langumir's theory of adsorption marks a significant turning point once in a hundred years (Swenson & Stadie, 2019). In spite of its simplicity, the Langumir adsorption equation sheds light on the basic physics of molecular interactions at surfaces and laid the foundation for later developments in engineering process design, adsorbent material development, and interface phenomena. The Langumir model has had an important impact on many different areas of chemical science, from materials science to chemical biology. With the development of better adsorption theories, this influence became much more noticeable and has continued until this day.

Often used to model Type I adsorption isotherms, the Langumir equation is one of the most effective adsorption isotherm equations (Afonso et al., 2016). The kinetic technique that Langumir initially employed for 2D monolayer surface adsorption was also used in their article to develop the equation's 1D equivalent, which can be applied in ultra micropores with single file diffusion systems.

The way reactants are adsorbed onto catalyst surfaces and how this affects reaction rates is clarified by the Langumir model of catalysis. This is essential to the design and optimization of catalytic processes. In environmental science, the Langumir equation can be used to model how pollutants adsorb onto materials like soil or activated carbon, emphasizing pollutant removal. To build adequate water and air filtration systems, this is required. In material science, the Langumir equation helps understand the interactions between different molecules and material surfaces when studying thin films and coatings.

The Langumir Blodgett and Langumir Boguslavski equations, two fundamental and exceedingly complex non-linear differential equations in engineering and basic sciences, were studied and solved analytically. The article analyzes these two non-linear differential equations using

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an unusual and uncomplicated technique that they tagged the Akbari-Ganji Method, or AGM (Hassanvand et al., 2023). AGM and numerical solution were compared in this research, and the findings show that this method is relatively simple and effective, making it suitable for use with other non-linear problems. The authors argued that there are significant benefits to this method of solving differential equations and that it can be used to solve a variety of sets of challenging differential equations that have not yet found workable solutions using other approaches.

On the other hand, the Van der Pol equation represents the behaviour of specific types of electrical circuits, particularly oscillators. It is especially useful for explaining circuits with non-linear resistors, such as vacuum tube oscillators and other non-linear electrical components. The Van der Pol equation is used in control systems to study the dynamics of non-linear damping systems. It can be used to evaluate the behaviour and stability of control systems that display complicated oscillatory phenomena, such as limit cycles. The Van der Pol equation is used to investigate rhythmic phenomena such as heartbeats and neural firing patterns and to model the activity of neurons and brain oscillators to better understand how rhythms are generated in biological systems.

First-order approximation perturbation was used to solve the Van der Pol differential equation.

$$x''(t) - \alpha\left(1 - x^2(t)\right)x'(t) + x(t) = 0.$$

Two different solutions were found. The forcing function that eliminated resonance was produced by the first solution, which limited the initial conditions to be $x(0)^{\,2} + x'^2 = 4$. This resulted in a stable solution; however, the initial conditions could only be around the origin of the phase plane space (Abbasi, 2024). The second solution allowed any initial conditions to be placed anywhere in the phase plane; nevertheless, the resulting forcing function produced resonance, making the system unstable over time. Phase plane graphs were used to compare the two solutions.

When the damping force is not proportional to the velocity, the Van der Pol equation can be used to describe mechanical systems with non-linear damping, such as vibrating systems or mechanical oscillators. When developing control strategies for mechanical systems and robots that exhibit non-linear dynamic behaviour, the equation is helpful in the field of robotics. The Van der Pol equation can explain oscillatory behaviour in chemical kinetics where non-linear influences are present, such as in biological reactions and enzyme kinetics.

The Van der Pol equation is a mathematical description of a second-order ordinary differential equation with cubic nonlinearities. Several studies incorporated time delay into the Van der Pol model (Elfouly & Sohaly, 2022). This study derives a delay differential equation from the Van der Pol model's differential equation and the resistor-inductor-capacitor (RLC) circuit. Because the Van der Pol delay model contains two delays, its applications can be reused in the proposed formula. In the case of minor delays, the Taylor series was used to obtain the ordinary differential equations from the delay differential equations.

The Van der Pol equation, which reflects economic dynamics' non-linear and oscillatory nature, has been used to predict economic cycles and fluctuations. On rare occasions, it is employed to model particular environmental phenomena, such as climate oscillations, where non-linear feedback mechanisms are responsible for periodic behaviour. The Van der Pol equation can be used to explain the dynamics of population systems that display oscillatory behaviour, where nonlinearity plays a significant role in population fluctuations.

The generalized Sundman transformation method was first used by Duarte et al. (1994), where only the Laguerre form of linearization was considered. Again, Mustafa et al. (2013) also examined the problem of linearization of non-linear second-order ODEs to the Laguerre form through the use of generalized Sundman transformations (S-transformations), as previously studied by Duarte et al. The outcomes acquired by Nakpim & Meleshko (2010) show that Duarte et al.'s use of the generalized Sundman transformation to solve the linearization problem for a second-order ordinary differential equation is insufficient.

Subsequently, Orverem et al. (2021b) affirmed what Nakpim & Meleshko had previously mentioned—that the generalized Sundman transformation (GST) was insufficient via the Laguerre form. The Emden differential equation was then linearized by the authors using this technique. The method (GST) was also used to solve the equations of Yang-Baxter by Orverem et al. (2021a). Another contribution uses the linearization approach to obtain the solutions to the modified Ivey's equation and the variable frequency oscillator equation. Two methods of linearization are examined: differential forms (DF) and the generalized Sundman transformation (GST) (Orverem & Haruna, 2023).

This research utilizes the non-Laguerre version of the generalized Sundman transformation to linearize the modified Langumir and Van der Pol second order non-linear ordinary differential equations.

METHOD

The Generalized Sundman Transformation (GST)

The formulae

$$u(t) = F(x,y),\ dt = G(x,y)dx,\ F_yG \neq 0, \qquad (1)$$

define a generalized Sundman transformation, which is a nonpoint transformation.

We need to find the necessary format for a second-order ordinary differential equation $y'' = f(x,y,y')$, that can be linearized to become a linear ordinary differential equation

$$u'' + \beta u' + \alpha u = \gamma. \quad (2)$$

The function u and its derivatives u' and u'' are defined by the first formula of equation (1), and its derivatives with respect to x gives

$$u'G = F_x + F_y y', \quad (3)$$

where $\frac{dt}{dx} = G(x,y)$ and $\frac{dy}{dx} = y'$.

Differentiation of equation (3) gives

$$u''G^2 + u'(G_x + G_y y') = F_{xx} + 2F_{xy}y' + F_{yy}y'^2 + F_y y'', \quad (4)$$

where $F_{xy} = F_{yx}$.

Substituting u' from (3) into (4) and simplifying, we have

$$u'' = \frac{G(F_y y'' + 2F_{xy}y' + F_{yy}y'^2 + F_{xx}) - (F_x G_x + F_x G_y y' + F_y G_x y' + F_y G_y y'^2)}{G^3}. \quad (5)$$

Next, by substituting equations (3) and (5) for u' and u'' respectively into equation (2) and simplifying, one has that

$$y'' + f_2 y'^2 + f_1 y' + f_0 = 0, \quad (6)$$

where

$$f_2 = \frac{(F_{yy}G - F_y G_y)}{K}, \quad (7)$$

$$f_1 = \frac{(2F_{xy}G - F_x G_y - F_y G_x + F_y \beta G^2)}{K}, \quad (8)$$

$$f_0 = \frac{(F_{xx}G - F_x G_x + F_x \beta G^2 + \alpha F G^3 - G^3 \gamma)}{K}, \quad (9)$$

and $K = GF_y \neq 0.$ Note from equation (2) that, $\alpha(t), \beta(t)$ and $\gamma(t)$ represent various functions.

Equation (6) is the necessary form of a second order ordinary differential equation that can be mapped into a linear equation (2) through a generalized Sundman transformation (1).

To obtain the sufficient conditions, one has to solve the compatibility system (7) to (9). One has to consider the system (7) to (9) as an overdetermined system of partial differential equations for the functions F and G with the coefficients $f_i(x,y)$, where $i = 0, 1, 2$.

The compatibility analysis depends on the value F_x. In this work, a complete solution is given for the case where $F_x = 0$.

From the system (7)-(9), one sees that

$$F_{yy} = \frac{f_2 G F_y + F_y G_y}{G}, \quad (10)$$

$$\beta = \frac{f_1 G + G_x}{G^2}, \quad (11)$$

and

$$\gamma = \frac{\alpha F G^2 - f_0 F_y}{G^2}. \quad (12)$$

Since $F_x = 0$, differentiating F_{yy} from equation (10) with respect to x and simplifying, on has that

$$G^2 f_{2x} + G G_{xy} - G_x G_y = 0. \quad (13)$$

Differentiating equation (11) with respect to x and simplifying gives

$$G_{xx} = \frac{2G_x^2 + G G_x f_1 - G^2 f_{1x}}{G}. \quad (14)$$

Differentiating equation (11) with respect to y and on simplification, it will result to

$$G_{xy} = G f_3 - G_y f_1, \quad (15)$$

where $f_3 = f_{1y} - 2f_{2x}$.

Next, differentiate equation (12) with respect to x and y respectively and simplify to have

$$2G_x f_0 - G f_{0x} = 0, \quad (16)$$

and

$$\alpha = \frac{G(f_0 f_2 + f_{0y}) - G_y f_0}{G^3}. \quad (17)$$

Now, substitute equation (15) into equation (13) to have

$$G G_y f_1 + G_x G_y - G^2(f_{2x} + f_3) = 0. \quad (18)$$

Comparing the mixed derivatives $(G_{xy})_x = (G_{xx})_y$ and simplifying, one can see that

$$G_x f_3 - G(f_{3x} + f_1 f_{2x} + f_{2xx}) = 0. \quad (19)$$

Differentiating α from equation (17) with respect to x and simplifying, one gets

$$2G_x(f_{0y} + f_0 f_2) + G_y(2f_0 f_1 + f_{0x}) - G(2f_0 f_3 + 4f_0 f_{2x} + f_{0xy} + f_{0x} f_2) = 0. \quad (20)$$

Differentiating α from equation (17) with respect to y and simplifying, becomes:

$$2G G_{yy} f_0 - 6G_y^2 f_0 + 2G G_y(2f_0 f_2 + 3f_{0y}) - G^2(f_4 + 2f_5 - f_1 f_3) = 0, \quad (21)$$

Where

$$f_4 = 2f_{0yy} - 2f_{1xy} + 2f_0 f_{2y} - f_{1y} f_1 + 2f_{0y} f_2 + 2f_{2xx},$$

and

$$f_5 = f_{2xx} + f_{2x} f_1 + f_{3x} + f_1 f_3.$$

If $f_3 \neq 0, f_5 \neq 0$ then from equation (19), we see that

$$G_x = \frac{G(f_{3x}+f_1 f_{2x}+f_{2xx})}{f_3}. \quad (22)$$

Substituting G_x into equation (16) and simplifying, we have that:

$$f_{0x} = \frac{2f_0(f_5 - f_1 f_3)}{f_3}, \quad (23)$$

where f_3 and f_5 are as previously defined.

Substituting G_x into equation (18) and simplifying, results to

$$G_y = \frac{G f_3(f_{2x}+f_3)}{f_5}. \quad (24)$$

Substituting G_x into equation (14), one differentiates equation (22) with respect to x and simplify to have as follows:

$$f_{2xxx} = f_3^{-1} f_5(f_5 + f_{3x}) - 2f_1 f_5 + f_{2x} f_1^2 + f_1^2 f_3 - f_{1x} f_3 - f_{2x} f_{1x} - f_{3xx}. \quad (25)$$

Next, substitute G_x into equation (15). Differentiating G_x from equation (22) with respect to y and simplifying gives:

$$f_{2xxy} = f_3^{-1} f_5 f_{3y} - f_1 f_{3y} - 2f_3 f_{2x} - f_{2xy} f_1 - 2f_{2x}^2 - f_{3xy}. \quad (26)$$

Substituting G_y from equation (24) into equation (21) and simplifying, we have:

$$f_3 f_5\left(6f_{0y} f_{2x} + 2f_{2xy} f_0 + 4f_{2x} f_0 f_2 + 2f_{3y} f_0 + 4f_0 f_2 f_3 + f_1 f_5\right) - f_3^2\left(6f_{2x}^2 f_0 + 12 f_{2x} f_1 f_3 - 6f_{0y} f_5 + 6f_0 f_3^2\right) - f_4 f_5^2 - 2f_5^3 = 0. \quad (27)$$

Linearization of the Modified Langumir and Van der Pol Differential Equations via GST

The original Langumir differential equation is given by

$$3yy'' + 3y'^2 + 4yy' + y^2 - 1 = 0. \quad (28)$$

Not all of the linearizability requirements were met by the coefficients of equation (28) using the GST as given here. As a result, we alter equation (28) to meet every linearizability requirement.

The modified Langumir equation is given as

$$3yy'' + 4y^2 y' + y^2 = 0 \quad (29)$$

To put equation (29) in the form of (6), we divide all through by $3y$ to have

$$y'' + \frac{4y}{3} y' + \frac{y}{3} = 0. \quad (30)$$

The coefficients of equation (30) are given as: $f_0 = \frac{y}{3}, f_1 = \frac{4y}{3}, f_2 = 0, f_3 = \frac{4}{3} \neq 0$, where $f_3 = f_{1y} - 2f_{2x}, f_4 = 2f_{0yy} - 2f_{1xy} + 2f_0 f_{2y} - f_{1y} f_1 + 2f_{0y} f_2 + 2f_{2xx}$ and $f_5 = f_{2xx} + f_{2x} f_1 + f_{3x} + f_1 f_3$.

That is,

$$f_4 = \frac{-16y}{9},$$

$$f_5 = \frac{16y}{9} \neq 0.$$

Testing the sufficient conditions, we have from equation (23) that

$$f_{0x} = 2f_0 \frac{(f_5 - f_1 f_3)}{f_3}.$$

That is, $0 = 0$.

Equation (26) becomes $0 = 0$. Equation (25) now becomes

$$0 = f_1^2 f_3^2 - 2f_1 f_3 f_5 + f_5^2, \quad (31)$$

that is,

$$0 = \frac{256y^2}{81} - \frac{512y^2}{81} + \frac{256y^2}{81} = \frac{512y^2}{81} - \frac{512y^2}{81} = 0.$$

Again, this condition is satisfied. Next, we check condition (27), which is reduced to

$$f_3 f_5(f_1 f_5) - f_3^2\left(-6f_{0y} f_5 + 6f_0 f_3^2\right) - f_4 f_5^2 - 2f_5^3 = 0. \quad (32)$$

That is

$$\frac{64y}{27}\left(\frac{64y^2}{27}\right) - \frac{16}{9}\left(\frac{-32y}{9} + \frac{32y}{9}\right) + \frac{4096y^3}{729} - \frac{8192y^3}{729} = \frac{4096y^3}{729} - \frac{16}{9}(0) + \frac{4096y^3}{729} - \frac{8192y^3}{729} = \frac{8192y^3}{729} - \frac{8192y^3}{729} = 0.$$

We can see that this equation can be linearized using the generalized Sundman transformation because all the necessary conditions are met. We already note that $F_x = 0$.

Next, we find equations (10), (22) and (24) as follows:

$$F_{yy} = \frac{F_y G_y + f_2 F_y G}{G} = \frac{F_y G_y}{G},$$

$$G_x = \frac{G(f_{2xx} + f_{2x} f_1 + f_{3x})}{f_3} = \frac{G(0)}{4/3} = 0,$$

and

$$G_y = \frac{G f_3(f_{2x}+f_3)}{f_5} = \frac{G}{y}.$$

We take the solution $F = y^2$ and $G = y$, which satisfies F_x, F_{yy}, G_x and G_y respectively. We therefore have from equation (1) that, $u = y^2,\ dt = ydx$.

To find α, β and γ, we have from equations (17), (11) and (12) that

$$\alpha = \frac{G(f_{0y} + f_0 f_2) - G_y f_0}{G^3} = \frac{y/_3 - y/_3}{y^3} = 0,$$

$$\beta = \frac{G_x + G f_1}{G^2} = \frac{4y^2/_3}{y^2} = \frac{4}{3},$$

and

$$\gamma = \frac{\alpha F G^2 - F_y f_0}{G^2} = \frac{-2y^2/_3}{y^2} = \frac{-2}{3}.$$

Therefore, $\alpha = 0,\ \beta = \frac{4}{3}$ and $\gamma = \frac{-2}{3}$ respectively.

Now, $u'' + \beta u' + \alpha u = \gamma$ (that is equation (2)) becomes

$$u'' + \frac{4}{3}u' = \frac{-2}{3},$$

or

$$3u'' + 4u' + 2 = 0.$$

The characteristics equation of the homogeneous equation is

$$r^2 + \frac{4}{3}r = 0.$$

That is

$$r\left(r + \frac{4}{3}\right) = 0 \implies r_1 = 0, r_2 = \frac{-4}{3}.$$

Therefore, the homogeneous solution is now

$$u_h = c_1 + c_2 e^{\frac{-4t}{3}}.$$

Using the method of variation of parameters, we have that for $n = 2$, and $u_h = c_1 + c_2 e^{\frac{-4t}{3}}$;

$$u_p = v_1 + v_2 e^{\frac{-4t}{3}}.$$

Since $y_1 = 1, y_2 = e^{\frac{-4t}{3}}$ and $\phi(t) = \frac{-2}{3}$, it follows that

$$v'_1 + v'_2(e^{\frac{-4t}{3}}) = 0,$$

$$v'_2\left(\frac{-4}{3}e^{\frac{-4t}{3}}\right) = \frac{-2}{3},$$

and

$$v'_2 = \frac{e^{\frac{4t}{3}}}{2}.$$

Substituting this into the first equation above we have;

$$v'_1 = \frac{-1}{2}.$$

Integrating, we have:

$$v_1 = -\int \frac{1}{2} dt = \frac{-t}{2}$$

and

$$v_2 = \frac{1}{2}\int e^{\frac{4t}{3}} = \frac{3}{8}e^{\frac{4t}{3}}.$$

Therefore

$$u_p = \frac{-t}{2} + \frac{3}{8},$$

and hence, the general solution is

$$u = c_1 + c_2 e^{\frac{-4t}{3}} - \frac{t}{2} + \frac{3}{8},$$

where $c_1,\ c_2$ are arbitrary constants.

We can now apply the GST to the equation as:

$$y^2 = c_1 + c_2 e^{\frac{-4t}{3}} - \frac{t}{2} + \frac{3}{8},\ \ t = \phi(x).$$

That is

$$y = \sqrt{c_1 + c_2 e^{\frac{-4\phi(x)}{3}} - \frac{\phi(x)}{2} + \frac{3}{8}},$$

where $t = \phi(x)$ is the solution of

$$\frac{dt}{dx} = (c_1 + c_2 e^{\frac{-4t}{3}} - \frac{t}{2} + \frac{3}{8})^{\frac{1}{2}}.$$

The original Van der Pol differential equation is given as

$$y'' - \mu(1 - y^2)y' + y = 0. \tag{33}$$

A survey reveals that the coefficients in the aforementioned equation are unable to satisfy all of the linearizable conditions, necessitating the modification of the equation to achieve our objective.

The modified Van der Pol differential equation we want to consider is given as

$$y'' + yy' + y = 0. \tag{34}$$

Equation (34) above is in the necessary form of equation (6) with the coefficients given as

$$f_0 = y,\ f_1 = y,\ f_2 = 0,$$

and

$$f_3 = 1 \neq 0,\ f_4 = -y,\ f_5 = y \neq 0.$$

Testing the sufficient linearizability conditions, we see that equation (23) becomes

$$f_{0x}f_3 = 2f_0(f_5 - f_1 f_3) = 2y(y - y) = 0.$$

Equation (25) is satisfied and becomes

$$f_{2xxx}f_3 = f_5(f_5) - 2f_1 f_3 f_5 + f_1^2 f_3^2 = y^2 - 2y^2 + y^2 = 0.$$

Equation (26) is also satisfied as

$$f_{2xxy} = f_3^{-1} f_5 f_{3y} - f_1 f_{3y} - 2f_3 f_{2x} - f_{2xy} f_1 - 2f_{2x}^2 - f_{3xy} = 0.$$

Finally, equation (27) is reduced to

$$f_3 f_5 (f_1 f_5) - f_3^2 (6f_0 f_3^2 - 6f_5 f_{0y}) - f_4 f_5^2 - 2f_5^3 = y(y^2) - 1(-6y + 6y) - (-y)y^2 - 2y^3$$

$$= y^3 + 6y - 6y + y^3 - 2y^3 = 0.$$

Now that all the sufficient conditions are satisfied, we proceed to determine F and G noting that $F_x = 0$, from equations (10), (22) and (24) to be:

$$F_{yy} = \frac{F_y G_y}{G},$$

$$G_x = 0,$$

and

$$G_y = \frac{G(0+1)}{y} = \frac{G}{y}.$$

Take $F = y^2$, $G = y$ and we see that, this solution satisfied all the conditions above. Now, equations (17), (11) and (12) becomes

$$\alpha = \frac{G(f_{0y} + f_0 f_2) - G_y f_0}{G^3} = \frac{y - y}{y^3} = 0,$$

$$\beta = \frac{G_x + G_2 f_1}{G^2} = \frac{y^2}{y^2} = 1$$

and

$$\gamma = \frac{\alpha F G^2 - F_y f_0}{G^2} = \frac{-2y^2}{y^2} = -2.$$

Therefore, $u'' + \beta u' + \alpha u = \gamma$ from equation (2) becomes

$$u'' + u' = -2,$$

or

$$u'' + u' + 2 = 0.$$

Characteristics equation of the homogeneous part is

$$r^2 + r = 0 \implies r(r+1) = 0$$

or

$$r_1 = 0, r_2 = -1.$$

Therefore, the homogeneous solution is

$$u_h = c_1 + c_2 e^{-t}.$$

Using the method of variation of parameters for $n = 2$ and $u_h = c_1 + c_2 e^{-t}$, we have:

$$u_p = v_1 e^0 + v_2 e^{-t}.$$

Since $y_1 = e^0$, $y_2 = e^{-t}$ and $\phi(t) = -2$, it follows that

$$v_1' + v_2'(e^{-t}) = 0,$$

$$v_2'(-e^{-t}) = -2,$$

and

$$v_2' = \frac{2}{e^{-t}}.$$

Substituting into the first equation above, we have;

$$v_1' = -2.$$

Integrating, we have;

$$v_1 = -\int 2dt = -2t,$$

and

$$v_2 = \int \frac{2}{e^{-t}} dt = 2\int e^t\, dt = 2e^t.$$

Therefore

$$u_p = -2t + 2,$$

and hence the general solution is now

$$u = u_h + u_p = c_1 + c_2 e^{-t} - 2t + 2,$$

or

$$u = c_1 + c_2 e^{-t} - 2t,$$

where $c_1 + 2 = c_1$ and c_1, c_2 are arbitrary constants.

Applying the generalized Sundman transformation, we have that

$$y^2(x) = c_1 + c_2 e^{-t} - 2t\,, \qquad t = \phi(x),$$

so that

$$y(x) = \sqrt{c_1 + c_2 e^{-\phi(x)} - 2\phi(x)},$$

where $t = \phi(x)$ is the solution of

$$\frac{dt}{dx} = \sqrt{c_1 + 2 + c_2 e^{-t} - 2t}.$$

CONCLUSION

This work uses the generalized Sundman transformation strategy to linearize the modified Langumir and Van der Pol non-linear second order ordinary differential equations. The conventional approach of variation of parameters is used to solve the linear equations that emerged from the linearization process. The two equations' solutions can be found by applying the Sundman transformation.

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