

ORIGINAL RESEARCH ARTICLE

Exploring Partially Ordered Multisets: Definitions, Applications, and Combinatorial Parameters

Funmilola Balogun^{1*}, Holyheavy. M. Balami², and Abdu Madugu³¹Department of Mathematics, Federal University Dutsin-Ma, Katsina, Nigeria²Department of Mathematics, Nigerian Army University, Biu, Kaduna, Nigeria³Department of Mathematics and Statistics, Umaru Musa Yaradua University, Katsina, Nigeria

ABSTRACT

A partially ordered set (S, \preceq) is a nonempty set S equipped with a partial order \preceq . Ordered structures are useful for representing application problems that involve comparable and incomparable parameters or inputs. Ordered sets are studied based on classical set theory, where objects in a collection are assumed to be distinct. However, mathematical objects are not always distinguishable, especially in applications. Multisets are mathematical models of entities with repetition. Multiset theory is distinguished from a set by carrying information on the number of times an element appears in a given collection, making it suitable for modeling real-life situations. A multiset M over S can be formally defined as a cardinal-valued function, $M: S \rightarrow \mathbb{N}$ such that $x \in \text{Dom}(M)$ implies $M(x)$ is a cardinal number and $M(x) = m_M(x) > 0$, where $m_M(x)$ denotes the number of times x occurs in M . Multisets generalize the classical sets and are apt for representing partial orders. There has been a growing interest in extending results on ordered sets to multisets. Unlike in classical set theory, there is still no unanimous way of studying foundational concepts in multisets. For instance, different approaches have been proposed for studying orderings on multisets due to the multiplicity parameter that is peculiar to multisets. This paper surveys studies on ordered multisets and compares existing orderings proposed on multiset structures. In particular, cases where the definition results in a partial order are of great interest in this exposition; their properties and theoretical implications make them suitable for application. We focus on definitions consistent with the Dershowitz-Manna multiset ordering, usually referred to as the standard multiset ordering, due to their relevance in applications, e.g., in computer programming. The strengths and possible limitations of the multiset orderings presented in this work are highlighted. This would aid in identifying potentially suitable definitions when dealing with studies that involve ordered multisets. Combinatorial parameters that have been studied on ordered multiset structures are also presented in this paper. The generalized notions of these parameters are investigated with some recommendations.

ARTICLE HISTORY

Received May 05, 2024

Accepted June 29, 2024

Published July 16, 2024

KEYWORDS

Multiset, partial order, ordered multiset, combinatorial parameters.

© The authors. This is an Open Access article distributed under the terms of the Creative Commons Attribution 4.0 License (<https://creativecommons.org/licenses/by-nc/4.0/>)

INTRODUCTION

Ordered structures are present in all aspects of mathematics. They occur in fields like graph theory and are useful for studying various algebraic structures (Jokela, 2023; Gosh, 2019; Caspard et al., 2012). Investigations on partially ordered sets remain a growing sphere of study, largely due to the many useful applications of ordered sets. For instance, ordered sets are employed in problems involving scheduling, ranking, and prioritization (Yap et al., 2021; Garmendia et al., 2017; Fattore, 2014; Blanchet-Sadri, 2012). They also have practical applications in decision-making (Muren et al., 2023; Muren et al., 2022; Qiao & Hu, 2020; Chen et al., 2013). Ordered set theory

assumes the axioms of the Cantorian set, and as such, the repetition of elements is insignificant.

A multiset extends the idea of a set by allowing multiple instances of its elements. This notion is more compatible with practical experiences. The study of multisets has both theoretical and practical implications (Gisin & Volkova, 2024; Alhazor et al., 2024; Liu et al., 2021; Felisiak et al., 2020; Jurgense, 2020; Singh et al., 2007). The range of a multiset M varies based on the theory under consideration. For most application problems, the image of any element in the set S is assumed to be in \mathbb{N} . However, in theoretical studies, the range can take values

Correspondence: Funmilola Balogun. Department of Mathematics, Federal University Dutsin-Ma, Katsina, Nigeria. ✉ fbalogun@fudutsinma.edu.ng.

How to cite: Balogun, F., Balami, H. M., & Madugu, A. (2024). Exploring Partially Ordered Multisets: Definitions, Applications, and Combinatorial Parameters. *UMYU Scientifica*, 3(3), 74 – 81. <https://doi.org/10.56919/usci.2433.009>

from any of \mathbb{Z} , \mathbb{R} , etc. (Felisiak et al., 2020). Multiset-like structures have been frequently used in applications - long before the concept was formalized. These structures have appeared under different names in the literature (e.g., heap, bag, bunch, finitely generated set). They have been rediscovered several times due to the practical need for models of structures that admit repetition. For instance, in combinatorial analysis, combinations with repetitions are common (Knuth, 1981; Kuba, 2022), also in membrane computing (Alhazor et al., 2024; Liu et al., 2021). Although classical sets and multisets are, in essence, at different abstraction levels, it is natural to explore a set theory-like framework to study the theory of multisets using set-theoretical foundations; Blizard's well-known results (Blizard, 1988) is a good foundational work on multisets (see also, Felisiak et al., 2020; Dang, 2014; Blizard, 1991). Since the introduction of multisets, there has been a growing interest in extending results from set theory to multisets. A partially ordered multiset (or pomset) generalizes an ordered set. As such, concepts and results from ordered set theory can be generalized to ordered multisets under suitable constructions and with the appropriate rules (Kuba, 2022; Balogun et al., 2020, 2021, 2022; Anusuya & Vimala, 2018; Balogun & Singh, 2017; Wilson, 2016; Girish and Sunil, 2009; Pratt, 1986). Ordered multisets have applications in fields such as computer science, economics, linguistics, and biology (Liu et al., 2021; Paun, 2010; Conder et al., 2007; Dershowitz & Ellerman, 2007). In computer science, for instance, multiset orderings are exploited in proving termination of programs and term rewriting systems, including production systems where programs are written in terms of rewrite rules (Martin, 1989; Dershowitz & Manna, 1979). Relative to ordered sets, the theory of ordered multisets is still under-explored. Different definitions have been proposed for comparing multisets or objects in a multiset using different approaches, with each approach embodying certain advantages and constraints. Some of these definitions are modifications or extensions of the standard multiset ordering (Dershowitz & Manna, 1979), and they are consistent with it. However, some other approaches differ completely from the Dershowitz-Manna definition. In this paper, we survey different proposals on multiset ordering with particular interest in definitions that result in a partial order. We also focus on definitions that are consistent with the standard multiset ordering due to their practical implications, these orderings are presented in section 2. The advantages and possible drawbacks of the orderings investigated are presented. By highlighting the strengths and limitations of these orderings via comparative analysis, this study aims to aid researchers working in related fields in identifying suitable orderings that can be adopted or modified for modeling problems that admit repeated elements. Multisets are valuable in combinatorics (Kuba, 2022; Balogun et al., 2020, 2021). In particular, partially ordered multisets are apt for modeling concurrent behaviors. Existing theories on partially ordered multisets are presented in section 3. Essential combinatorial objects, such as the height and width of a partially ordered multiset,

are presented in the next section. Several characterizations exist for these parameters. Linear extensions, realizers, and dimensions of ordered sets are significant combinatorial concepts with practical applications. Analogous forms of these combinatorial parameters have been proposed for ordered multisets. For instance, the notion of linear extension has been studied on partially ordered multiset structures (Balogun et al., 2022; Singh et al., 2012). These generalized combinatorial parameters need further vindication as they could efficiently model problems involving scheduling, ranking, and multi-criteria decision-making where repetition is significant.

ORDERINGS ON MULTISSETS

In this section, we review orderings proposed on multisets with some examples. The similarities and peculiarities, strengths and limitations of these multiset orderings are also discussed.

Dershowitz-Manna Multiset Ordering

Dershowitz & Manna (1979) presented a multiset ordering by defining multisets over well-founded sets. In their definition, the well-founded set S is assumed to induce an ordering on $\mathbf{M}(S)$ i.e., the class of finite multisets defined on S . This ordering is defined as follows:

For $A, B \in \mathbf{M}(S)$, $A \ll B$ if $\exists X, Y \in \mathbf{M}(S)$ such that,

- i. $\emptyset \neq X \subseteq B$
- ii. $A = (B \setminus X) + Y$, and
- iii. X dominates Y , that is, for all $y \in Y$, there is some $x \in X$ such that $y \prec x$.

This ordering is usually referred to as the standard multiset ordering and often serves as a reference for other multiset orderings - especially for orderings introduced to prove termination of programs and term rewriting systems. The ordering is well-founded, provided the ordering on the generic set is well-founded.

Example:

Let $A, B \in \mathbf{M}(S)$ where

$$A = \{2/a_1, 4/a_2, 2/a_3, 4/a_4\}$$

$$B = \{2/a_1, 2/a_3, 6/a_4, 2/a_5\}$$

$$X = \{2/a_4, 2/a_5\}$$

$$Y = \{4/a_2\} \quad (n/a_i \text{ implies the object } a_i \text{ occurs in the multiset under consideration } n \text{ times}).$$

Suppose $a_i \prec a_j \Leftrightarrow i < j$.

Since conditions i - iii above are satisfied, $A \ll B$

The Dershowitz-Manna ordering is monotonic, a desirable property, especially in application. Jouannaud and Lescanne (1982) observed that it is, so far, the strongest monotonic ordering that has been defined for these structures. A major drawback of this ordering is its

limitation in describing the inclusion relation between two multisets; a different definition would have to be adopted to achieve this.

An important application of the Dershowitz-Manna multiset ordering is proving the termination of program and term rewriting systems.

Huet-Oppen Multiset Ordering

Building on the Dershowitz-Manna definition, a more tractable multiset ordering was proposed by Huet & Oppen (1980) as follows:

For any $A, B \in M(S)$, $A \ll B \Leftrightarrow A \neq B$ and $[A(y) > B(y) \Rightarrow (\exists x \in S)x \succ y \ \& \ A(x) < B(x)]$

Example:

Let $A = \{2/a_1, 3/a_2, a_3, 2/a_4\}$ and $B = \{a_1, a_2, 2/a_3, 6/a_4\}$

Suppose $a_i < a_j$ whenever $i < j$. Clearly $A \ll B$.

This ordering is also induced by the ordering $<$ on the ground set S . It is well-founded if and only if $(S, <)$ is well-founded. The ordering is linear and becomes a lexicographic ordering if $<$ on the set S is total. This multiset ordering also has applications in establishing program termination and in term rewriting systems. This ordering is equivalent to the Dershowitz-Manna multiset ordering.

Jouannaud-Lescanne Multiset Ordering

Another efficient multiset ordering was proposed by Jouannaud & Lescanne (1982). These authors extended Dershowitz and Manna’s ordering via two definitions. In the first definition, $A \ll_1 B \Leftrightarrow \bar{A} \ll_1^{lex} \bar{B}$. Where the following holds for the partition $\bar{A} = \{A_k | k = 1 \dots p\}$:

- i. $t \in A_k \Rightarrow A_k(t) = A(t)$.
- ii. $t \in A_k$ and $s \in A_k \Rightarrow t$ and s are incomparable.
- iii. $\forall k \in [2, \dots, p] t \in A_k \Rightarrow (\exists s \in A_{k-1}) s \succ t$.

Example:

If $a_1 < a_2$ and $A = \{a_1, a_1, a_2, 3\}$ then $A_1 = \{3, a_2\}, A_2 = \{a_1, a_1\}$. If $B = \{a_1, a_2, a_2, a_1, 3, 3, 3\}$ then $B_1 = \{3, 3, 3, a_2, a_2\}, B_2 = \{a_1, a_1\}$ and $A \ll_1 B$.

Clearly, \ll_1 is an ordering because lexicographic extension preserves it. These authors proved that the ordering \ll_1 contains the Dershowitz and Manna ordering, whereas the converse is false.

On the other hand, the ordering \ll_1 has a strong limitation; it is not monotonic. Hence it cannot be employed when the initial ordering is increased.

The second definition uses another approach to building the partition. Here, the construction puts elements that

are the same in different multisets of the partition. $A \ll_2 B \Leftrightarrow \bar{A} \ll_2^{lex} \bar{B}$ and the following hold for $\bar{A} = \{A_k | k = 1 \dots p\}$:

- i. A_k is a set that is $A_k(t) \leq 1$.
- ii. $t \in A_k$ and $s \in A_k \Rightarrow t$ and s are incomparable.
- iii. $\forall k \in [2, \dots, p] t \in A_k \Rightarrow (\exists s \in A_{k-1}) s \succ t$

The two definitions differ only in the first condition. This ordering contains the Dershowitz and Manna multiset ordering, but the converse is false. Using their definitions, the authors presented an efficient implementation of Dershowitz and Manna’s ordering. Also, the Jouannaud-Lescanne definition produces a well-founded ordering provided $>$ on the ground set has this property. An important application of this multiset ordering is defining combinatorial parameters in the multiset setting (Balogun et al., 2021, 2022).

Krom Multiset Ordering

Also building on the Dershowitz-Manna multiset ordering, Krom (1985) defined two multiset orderings together with their respective transitive closures as follows:

For multisets A and B defined over a set X , $A \ll^n B$ if and only if

- i. $\exists! y \in X$ with $A(y) < B(y)$
- ii. $A(y) + 1 = B(y)$ for the one element in i above
- iii. if $A(x) > B(x)$ then $x < y$ in X
- iv. $\sum(A(x) - B(x)) \leq n$, the sum is over all $x \in X$ such that $A(x) > B(x)$

For \ll^∞ , if condition iv is omitted, then $A \ll^\infty B$.

An extension of the partial order $<$ on X is the ordering. $<_\epsilon$ Defined thus:

$A <_\epsilon B$ if and only if there exists $x \in X$ such that $B(x) > 0$ and for any $y \in X$, if $A(y) > 0$, then there exists $z \in X$ such that $y < z$ and $B(z) > 0$ i.e.,

$$B \neq \emptyset \wedge \forall y \in A \rightarrow \exists z \in B \wedge y < z.$$

The defined ordering inherits the well-founded property if it exists on the underlying set $(X, <)$, making it useful for application in computer programs.

Martin Multiset Ordering

Martin (1989) presented a definition via a geometric approach. Martin’s definition is a generalization of Dershowitz-Manna ordering using matrix action. Martin exploited the notion of cone in \mathbb{R}^n . The author showed that tame orderings (i.e., an ordering on a multiset that preserves both multiset union and an ordering on the ground set) can be classified geometrically. He also

showed that this method can be employed in presenting a unification of different orderings on multiset structures.

The following definition was given:

For two multisets A and B defined on S , if $<$ is a strict order on S and K an n by n matrix over \mathbb{N} indexed by the elements of S , then $A \ll B \Leftrightarrow f_x(A) \leq f_x(B) \forall x \in S$ where $f_x(B) = \sum_{y \geq x} B(y)$

Martins' multiset ordering is well-founded and monotonic. The proposed ordering embodies properties that arise naturally in termination proofs. Hence, it is useful in proofs of program termination and in equational reasoning algorithms based on term rewriting systems. In their classification, the standard multiset ordering (or an ordering got from it by the action of an invertible matrix) is shown to be the strongest tame ordering defined on the class $\mathbf{M}(S)$ of finite multisets on S .

Savaglio-Vannucci Multiset Ordering

The following multiset ordering was proposed in Savaglio & Vannucci (2007):

Let $A, B \in \mathbf{M}(S)$, then $A \gg B \Leftrightarrow A(x) > B(x) \forall x \in S$, i.e., the multiset A dominates the multiset B if and only if B is a proper submultiset of A .

In this case, the ordering \gg does not depend on the underlying ordering on S . This ordering cannot be total as the inclusion relation on $\mathbf{M}(S)$ is not total. Also, the different properties of the underlying base set, such as well-foundedness, do not naturally extend to the elements of $\mathbf{M}(S)$ via this definition.

PARTIALLY ORDERED MULTISSETS (POMSETS)

Partially ordered multisets are useful for representing concurrent behaviors. Through different notions of comparability/incomparability, various partially ordered multiset structures have been proposed for modeling concurrency (Balogun et al., 2021, 2022; Girish & Sunil, 2009; Pratt, 1986; Gischer, 1984; Mazurkiewicz, 1984). An extensive study of equational theories of ordered multisets, sets of ordered multisets, ideals of ordered multisets, and how they relate to theories of languages is presented in Gischer (1984). Gischer's work appears to be, so far, the most comprehensive work on the algebraic theory of ordered multisets. He examined some theories and established whether they are finitely axiomatizable for each equational theory of ordered multisets and languages under concatenation, concurrence, and other operations that can be defined on an ordered multiset structure.

A natural generalization of language as a set of strings is the notion of a set of ordered multisets as a process. A string is an ordered set of symbols, or preferably multiset; considering the fact that repetition of symbols is permitted. Quite a number of formal language theory

intuitions generalize easily when the total order is relaxed to a partial one. The model of a partial string, called a *partial word*, as a generalization of formal language theory was initiated by Grabowski (1981). This generalization of a string is called a pomset (the term partially ordered multiset or pomset together with an in-depth characterization was proposed in Pratt, 1986). Via a formulation that exploits the notion of a partial string (obtained by relaxing the linearity constraint on a string), Pratt (1986) presented a single hybrid approach for expressing concurrency. A combination of formal languages, partial orders, and temporal logic was used to propose a rich language that blends logic and algebra, giving rise to a natural class of models of concurrent processes. The method employed is direct and involves fewer artificial constructs- an advantage over some of the existing concurrency models- and still applies to various systems. Various effective operations for formally defining concurrent processes were introduced and categorized under combinatorial, Boolean, and homomorphisms. Some of the operations proposed by Pratt correspond to familiar programming language constructs, while others require further vindication.

Some concepts and definitions proposed for pomsets are presented below.

Definition: A structure $(V, \preceq, \sigma, \Sigma)$ is called a *labeled partial order* (lpo) over a set Σ where \preceq is a partial order on V and $\sigma: V \rightarrow \Sigma$ maps each $v \in V$ to an element of Σ .

Note that Σ is an alphabet of actions and V denotes instances of that alphabet, or events that form a word, where \preceq represents the order of occurrences of letters in the word. The usual formal language theoretical notion of a word obtains. If \preceq is a total order, we get the regular formal language theoretical concept of a word. In an atomic *labeled partial order* the cardinality of V is 1.

An isomorphism of labeled partial orders is a map (f, t) such that f is an isomorphism of posets and the identity map on Σ is t . A map (f, t) where t is the identity and f is the identity function on the elements of the ordered set (but not necessarily an isomorphism of posets) is called an *augmentation* of labeled partial orders. An augmentation yields an augment of its argument.

Definition: The isomorphism class of a labeled partial order is called a *pomset*.

A pomset can be informally viewed as a labeled partial order where attention is paid to the selection of the set V , instead of its size, but other information are maintained. Hence, replacing $V = \{0,1,2\}$ by $V = \{5,6,7\}$ while retaining σ or \preceq does not alter the pomset.

Rensik (1996) describes a pomset as node-labelled directed graphs where the edge-relation is irreflexive and transitive,

interprets up to label and edge-preserving isomorphism such that the identity of the nodes is abstracted from them, while their ordering is preserved. His investigation is around a particular class of pomset called *order-deterministic*. This class of pomsets has as a proper subset all ordered sets. Order-deterministic pomsets form a distributive lattice under pomset prefix. Among other attributes, order-deterministic pomsets give rise to a reflexive subcategory of the category of all pomsets.

In Girish & Sunil (2009), multiset relations are introduced as follows:

Definition. Let A and B be two multisets drawn from a set S ; then $A \times B$ (*Cartesian product*) is given by

$$A \times B = \{(m/x, n/y)/mn; x \in^m A, y \in^n B\}$$

A submultiset R of $A \times A$ is a *multiset relation* on A if every member $(m/x, n/y)$ of R has count $C_1(x, y) \cdot C_2(x, y)$. We write $m/xRn/y$ whenever m/x is related to n/y .

The domain ($Dom R$) and range ($Ran R$) of the relation R are given by the following sets, respectively:

$$\{x \in^r A; \exists y \in^s A \text{ such that } r/xRs/y\},$$

$$\{y \in^s A; \exists x \in^r A \text{ such that } r/xRs/y\}$$

Based on these relations, a pomset is defined as a multiset with a reflexive, antisymmetric and transitive relation. The authors proposed different ways of ordering a multiset with respect to the defined relations.

Balogun & Tella (2018) proposed a partial ordering on a multiset structure with a non-trivial involvement of the multiplicity. The ordering on the set of multiplicities is assumed to be a partial order instead of the usual natural ordering. The authors defined a pomset as follows:

Let (S, \preceq) and (\mathbb{N}, \preceq) be ordered sets such that \mathbb{N} is also partially ordered. For any two points m_i/s_i and m_j/s_j (where m_i represents the multiplicity of the object s_i) in $M \in \mathbf{M}(S)$, we have $m_i/s_i \preceq m_j/s_j \Leftrightarrow ((s_i \preceq s_j) \wedge (m_i \leq m_j))$. The two points coincide if and only if $s_i = s_j$ ($m_i = m_j$ follows from the exact multiplicity axiom of MST in Blizard, 1988). The two points are comparable if and only if $(m_i/s_i \preceq m_j/s_j) \vee (m_j/s_j \preceq m_i/s_i)$ otherwise they are incomparable.

It is easily deducible from the above definition that, given any of the conditions below, a point m_i/s_i cannot precede another point m_j/s_j

- i. $m_i/s_i \not\preceq m_j/s_j$
- ii. $m_i/s_i \not\preceq m_j/s_j$
- iii. $m_i/s_i \not\preceq m_j/s_j$

The definition gives rise to a new concept called *semimset chains*. A semimset chain is obtained whenever any two objects of the underlying ordered set or their multiplicities are comparable. The proposed definition needs further

vindication as it promises to be useful for modeling complex systems.

SPECIAL INSTANCES AND COMBINATORIAL PARAMETERS OF A PARTIALLY ORDERED MULTISSET

Some special cases of pomsets are described in this section (see Rensink, 1996). We also discuss combinatorial parameters that have been generalized to ordered multisets.

Mazurkiewicz traces: Also known as partially commutative monoids, Mazurkiewicz traces consist of strings and multisets. Here, the set S under consideration (also known as a concurrent alphabet) has the structure $(S, <)$ where $< \subseteq S \times S$ is an irreflexive and symmetric independency relation. Mazurkiewicz traces and the pomset model studied by Pratt (1986) are somewhat closely related. The independency relation determines the degree to which the concatenation operator \odot is commutative. The traces are freely generated by $\Sigma_{maz} = \langle \varepsilon, \odot, < \rangle$ where ε is the identity element and \odot an associative operation, satisfying the following:

$$\begin{aligned} \vdash \varepsilon < a & \quad (1) \\ a < b \vdash b < a & \quad (2) \\ a < b, a < c \vdash a < (b \odot c) & \quad (3) \\ a < b \vdash a \odot b = b \odot a & \quad (4) \end{aligned}$$

The above system is a model for strings if $<$ is the empty order. The system is a model for multisets whenever $\preceq = (S \times S) \setminus \{(a, a) | a \in S\}$.

Series-parallel pomsets: Consists of a direct combination of the algebras of strings and multisets, where the neutral element of a string is the same as that of a multiset. Series-parallel pomsets are also known as N -free pomsets (Gischer, 1988; Pratt, 1986). This is probably the most intensively studied approach for obtaining an extensive theory of pomsets. The signature freely generates series-parallel pomsets $\Sigma_{sp} = \langle \varepsilon, \cdot, \uplus \rangle$ with these operations:

$$\begin{aligned} \varepsilon \cdot a = a & \quad (1) \\ a \cdot \varepsilon = a & \quad (2) \\ (a \cdot b) \cdot c = a \cdot (b \cdot c) & \quad (3) \\ \varepsilon \uplus a = a & \quad (4) \\ (a \uplus b) \uplus c = a \uplus (b \uplus c) & \quad (5) \\ a \uplus b = b \uplus a & \quad (6) \end{aligned}$$

Only N -free models (probably not all N -free models) can be generated via this signature e.g. the empty and singleton pomsets.

Forests: These pomsets (also known as multisets of trees) possess a special attribute that all predecessors of a given element are linearly ordered. They are viewed as an algebraic extension of multisets with an associative concatenation operator (denoted by $;$) for which the empty forest is left cancellative, instead of left and right

neutral obtainable for strings, and distributes over addition from the right. They are freely generated by the signature $\sum_{tr} = \langle \varepsilon, ;, \cup \rangle$ using operations 4, 5, and 6 combined with the axioms below:

$$\varepsilon; a = a \quad (7)$$

$$(a; b); c = a; (b; c) \quad (8)$$

$$(a \cup b); c = a; c \cup b; c \quad (9)$$

The above algebra is presented with the additional axiom $a \cup a = a$ in Beaten and Weijland (1990). This system is a model for pomsets that are hierarchical orders with termination. Forest addition coincides with multiset addition.

Order-deterministic pomsets. These pomsets are constructed by replacing pomset union with a new operator \sqcup called join. Order-deterministic pomsets are generated by the signature $\sum_{det} = \langle \varepsilon, \cdot, \sqcup \rangle$ with operations 1, 2 and 3 of series parallel pomsets together with:

$$\varepsilon \sqcup a = a \quad (10)$$

$$(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c) \quad (11)$$

$$a \sqcup b = b \sqcup a \quad (12)$$

$$a \cdot (b \sqcup c) = (a \cdot b) \sqcup (a \cdot c) \quad (13)$$

Mazurkiewicz traces are properly contained in order-deterministic pomsets.

Combinatorial Parameters

Combinatorial parameters that have been studied on ordered multiset structures are presented in this section.

The notions of homomorphism and substitution are well-established in formal language theory. Both notions have been generalized from strings to pomsets. Homomorphism in formal language theory is defined on sets of pomsets, whereas the same notion as a mapping defined on pomsets goes between single pomsets.

As noted in section 2, the multiset ordering presented in Jouannaud & Lescanne (1982) is apt for defining combinatorial parameters in the multiset setting. The multiset ordering that employs set-based partitioning was adopted in Balogun et al. (2021) to construct sets of incomparable elements of a pomset. With this construction, the authors generalized results on antichains from ordered set theory to multisets. The method of set-based partitioning guarantees that in any antichain, no two elements are comparable or equal (since elements are allowed to repeat) under the defined order.

Linear extension, realizer and dimension of ordered sets have been extended to multisets. These combinatorial parameters are useful in application problems such as the scheduling problem also known as the jump number problem.

Linear extensions

A linear argument of an ordered multiset p is a *linearization* of p . The set containing linearizations of an ordered multiset is usually denoted by $\lambda(p)$. This extends to $\lambda(P)$ where P is a set of pomsets and $\lambda(P) = \bigcup_{p \in P} \lambda(p)$.

The method of constructing a linearization, also known as linear extension, of an ordered structure is key in determining an optimal realizer for the structure.

In Balogun et al. (2022), linear extensions were studied on a partially ordered multiset structure. The extension theorem was generalized under suitable constraints. Also, a heuristic algorithm that constructs multiset linear extensions of a finite ordered multiset was presented. This study can be extended to multisets in the class of finite multisets $\mathbf{M}(S)$. Singh et al. (2012) presented an application of topological sorting to obtain linear extensions of a partially ordered class of finite multisets. The multisets used are built from a partially ordered base set. The authors established that the partially ordered class of finite multisets is an intersection of its linear extensions (i.e., the realizer of the ordered multisets)

Dimension

The dimension of an ordered structure is an important combinatorial parameter and has several characterizations (Joret et al., 2016). This notion has been generalized using ordered multiset structures. For instance, the dimension of an ordered multiset p can be described as the dimension of the base poset of a representative labeled partial order of p .

In Balogun and Singh (2017), the dimension of a pomset was studied relative to the dimension of the underlying poset. Among other findings, the authors established that the dimension of a pomset is bounded by the dimension of the generating poset.

The generalized combinatorial parameters presented in this study are valuable in applications, especially in modeling problems where repetition is significant. Studies on decision-making, for instance, that are based on set theory can be extended via these generalized parameters. These combinatorial parameters need further exploration using multiset structures for more efficient outcomes.

CONCLUSION

Different orderings proposed on multiset structures were studied in this paper, with a focus on definitions that are consistent with the standard multiset ordering. These orderings have applications in areas such as computer programming. They are particularly useful in proving program termination. The strengths and limitations of the proposed orderings were investigated via a comparative analysis. The findings presented will aid in identifying suitable orderings that can be adopted for modeling problems involving repeated elements. The study of combinatorial parameters of an ordered multiset is a

growing field with possible applications in areas such as scheduling, ranking, and decision-making. Extensions, realizer and dimension of an ordered multiset have been defined via existing multiset orderings. These combinatorial parameters can be further generalized by modifying or extending existing theories of partially ordered multisets, especially based on the multiplicity parameter.

REFERENCES

- Alhazor, A., Freund, R., Ivanor, S. et al. (2024). P systems with reactive membranes. *Journal of Membrane Computing*, 6: 82-93. [\[Crossref\]](#)
- Anusuya Ilamathi, V. S., & Vimala, J. (2018). Multi-criteria decision making on lattice ordered multisets. In: Thampi, S., Mitra, S., Mukhopadhyay, J., Li, K.C., James, A., Berreti, S. (eds) Intelligent Systems Technologies and Applications. ISTA 2017. Advances in Intelligent Systems and Computing, vol 683. Springer, Cham. [\[Crossref\]](#)
- Balogun, F., & Singh, D. (2017). Some characterizations for the dimension of ordered multisets. *FUDMA Journal of Sciences*, 1(2), 84-87.
- Balogun, F., Singh, D., & Aliyu, S. (2022). Multisets linear extensions with a heuristic algorithm, *Annals of Fuzzy Mathematics and Informatics*, 24(2), 129-136.
- Balogun, F., Singh, D., & Tella, Y. (2020). Realizers of partially ordered multisets, *Theory and Applications of Mathematics and Computer Science*, 10(2), 1-6.
- Balogun, F., Singh, D., & Tella, Y. (2021). Maximal and maximum antichains of ordered multisets, *Annals of Fuzzy Mathematics and Informatics*, 21(1), 105-112.
- Balogun, F., & Tella, T. (2018). A study on some substructures of ordered multisets, *FUDMA Journal of Sciences*, 2 (1), 201-205.
- Beaten, J. C. M., & Weijland, W. P. (1990). *Process Algebra*. Cambridge University Press, Cambridge. [\[Crossref\]](#)
- Blanchet-Sadri, F. (2012). Algorithmic combinatorics of partial words. *International Journal of Foundations of Computer Science*, 1189-1206. [\[Crossref\]](#)
- Blizard, W. (1988). Multiset theory. *Notre Dame Journal of Formal Logic*, 30(1), 36-66. [\[Crossref\]](#)
- Blizard, W. (1991). The development of multiset theory. *Mod. Log*, 1(4), 319-352.
- Caspard, N., Leclerc, B., & Monjardet, B. (2012). *Chains and antichains*. In: Finite Ordered Sets: Concepts, Results and Uses. Encyclopedia of Mathematics and its Applications. Cambridge University Press: 107-129. [\[Crossref\]](#)
- Chen, S., Liu, J., Wang, H., & Augusto, J. C. (2013). Ordering based decision making – A survey. *Information Fusion*, 14(4): 521-531. [\[Crossref\]](#)
- Conder, M., Marshall, S., & Slinko, A. (2007). Orders on Multisets and Discrete Cones. *Order*, 24, 277-296. [\[Crossref\]](#)
- Dang, H. (2014). A single-sorted theory of multisets. *Notre Dame Journal of Formal Logic*, 55(3) (2014), 299-332. [\[Crossref\]](#)
- Dershowitz, N., & Ellerman, E. C. (2007). Leanest quasi-orderings. *Information and Computation*, 205(4): 535-556. [\[Crossref\]](#)
- Dershowitz, N., & Manna, Z. (1979). Proving termination with multiset orderings. *Communications of the Journal of Association for Computing Machinery*, 22(8), 465-476. [\[Crossref\]](#)
- Fattore, M. (2014). Partially ordered sets. In: Michalos, A. C. (eds) Encyclopedia of Quality of Life and Well-being Research. Springer. Dordrecht. [\[Crossref\]](#)
- Felisiak, P. A., Qin, K., & Li, G. (2020). Generalized multiset theory. *Fuzzy Sets and Systems*, 380, 104-130. [\[Crossref\]](#)
- Garmendia, L., Gonzalez del Cumpo & Recusens, J. (2017). Partial orderings for hesitant fuzzy sets. *International Journal of Approximate Reasoning*, 84: 159-167. [\[Crossref\]](#)
- Girish, K. P., & Sunil, J.J. (2009). General relations between partially ordered multisets and their chains and antichains. *Mathematical Communications*, 14(2), 193-205.
- Gischer, J. (1984). Partial orders and the axiomatic theory of shuffle. PhD Thesis, Computer Science Department, Stanford University.
- Gischer, J. (1988). The equational theory of pomsets. *Theoretical Computer Science*, 61, 199-224. [\[Crossref\]](#)
- Gisin, V., & Volkova, E. S. (2024). Equivalence relations on Multisets. 2024 XXVII International Conference on Soft Computing and Measurements (SCM). Saint Petersburg, Russian Federation. [\[Crossref\]](#)
- Gosh, A. (2019). A study of some combinatorial sets in partial semigroups. *Asian-European Journal of Mathematics*. [\[Crossref\]](#)
- Grabowski, J. (1981). On partial languages. *Fundamenta Informaticae*, 4, 427-498. [\[Crossref\]](#)
- Huet, G., & Oppen, D. C. (1980). *Equations and rewrite rules: a survey*, in R. Book, ed., Formal Languages Perspectives and Open Problems, Academy Press, New York. [\[Crossref\]](#)
- Jokela, J. (2023). General mixed lattices. *Order*. [\[Crossref\]](#)
- Joret, G., Micek, P., Milans, K., Trotter, W. T., Walczak, B., & Wang, R. (2016). Tree-width and dimension. *Combinatorica*, 36, 431-450. [\[Crossref\]](#)
- Jouannaud, J-P., & Lescanne, P. (1982). On multiset orderings. *Information Processing Letters*, 15(2), 57-63. [\[Crossref\]](#)
- Jurgense, H. (2020). Multisets, heaps, bags, families: What is a multiset? *Mathematical Structures in Computer Science*, 30: 139-158. [\[Crossref\]](#)
- Knuth, D. E. (1981). *The Art of Computer Programming (Seminumerical Algorithms)*, volume 2. Addison-Wesley, Reading, Massachusetts, second edition.
- Krom, M. (1985). Partial ordering for sets of multisets. *Algebra Universalis*, 21, 156-162. [\[Crossref\]](#)

- Kuba, M. (2022). On multisets, interpolated multiple zeta values and limit laws. *The Electronic Journal of Combinatorics*, 29(1): #P1.48. [\[Crossref\]](#)
- Liu, Y., Nicolescu, R., & Sun, J. (2021). An efficient labeled nested multiset unification algorithm. *Journal of Membrane Computing* 3, 194-204. [\[Crossref\]](#)
- Martin, U. (1989). A geometrical approach to multiset orderings. *Theoretical Computer Science*, 67, 37-54. [\[Crossref\]](#)
- Mazurkiewicz, A. (1984). Traces, Histories, Graphs: Instances of a process monoid. *Proceedings of the Conference on Mathematical Foundations of Computer Science, Springer-Verlag LNCS*, 176. [\[Crossref\]](#)
- Muren, Liu, C., & Cui, W. (2023). The relationships among group decision-making units based on partial ordered set. *Computers and Industrial Engineering*, 179. [\[Crossref\]](#)
- Muren, Liu, C., & Cui, W. et al., (2022). Special relationship among decision-making units based on partially ordered set and new evaluation and projection methods. *Journal of Systems Science and Systems Engineering*, 31(1): 226-246. [\[Crossref\]](#)
- Paun, G. (2010). A quick guide to membrane computing. *The Journal of Logic and Algebraic Programming*, 79: 291-294. [\[Crossref\]](#)
- Pratt, V. R. (1986). Modelling concurrency with partial orders. *International Journal of Parallel Programming*, 15, 33-71. [\[Crossref\]](#)
- Qiao, J., & Hu, B. Q. (2020). On decision evaluation functions in generalized three-way decision spaces. *Information Sciences*, 507: 733-754. [\[Crossref\]](#)
- Rensink, A. (1996). Denotational, causal, and operational determinism in event structures, in *Trees in Algebra and Programming- CAAP '96*, edited by H. Kirchner. *1059 Lecture Notes in Computer Science, Springer-Verlag, Berlin*. [\[Crossref\]](#)
- Savaglio, E., & Vannucci, S. (2007). On multidimensional inequality in partitions of multisets. *Quaderno de Dipartimento di Economica Politica N504. University of Siena*.
- Singh, D., Ibrahim, A.M., Yohanna, T., & Singh, J.N. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, 37(2): 73-92.
- Singh, D., Tella, Y., & Singh, J. N. (2012). Topological sorts of a multiset ordering. *IJCSST*, 5(2): 101-105.
- Wilson, A. T. (2016). An extension of MacMahon's equidistribution theorem to ordered multiset partitions. *The Electronic Journal of Combinatorics*, 23(1) #P1.5 [\[Crossref\]](#)
- Yap, K. T., Wehlan, D., & Zaguia, I. (2021). Permutations Avoiding Certain Partially-Ordered Patterns. *The Electronic Journal of Combinatorics*. [\[Crossref\]](#)