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ORIGINAL RESEARCH ARTICLE

Bayesian and Maximum Likelihood Estimation of the Weibull-Power Function Scale Parameter: A Loss-Function Comparison Study

Hauwa Abdulrahman Dangana[1], Abubakar Usman[2], Jamilu Garba[2] and Ibrahim Abubakar Sadiq[2]
[1]Department of Public Health, Intercountry Centre for Oral Health (ICOH) for Africa, Jos, Nigeria
[2]Department of Statistics, Ahmadu Bello University, Zaria, Nigeria

ABSTRACT

This study examines the estimation of the scale parameter of the Weibull-Power Function Distribution (WPFD) using both maximum likelihood and Bayesian approaches. Bayesian estimation is conducted under one informative Gamma prior with hyperparameters (a, b), as well as two non-informative priors: the uniform prior and Jeffreys' prior. For each prior specification, Bayes estimators are derived under squared error, quadratic, and precautionary loss functions. Closed-form expressions for the posterior distributions and corresponding Bayes estimators are obtained. The finite-sample performance of the competing estimators is evaluated through a Monte Carlo simulation study based on 1000 replications. Estimator performance is assessed using mean squared error (MSE), bias, and coverage probability. The results indicate that all estimators are consistent, with bias and MSE decreasing as sample size increases. Across different prior specifications and parameter settings, the Bayesian estimator under the quadratic loss function consistently attains the lowest MSE, yielding reductions of approximately 10-20% relative to the maximum likelihood estimator in small and moderate samples. These findings suggest that Bayesian estimation under quadratic loss provides improved finite-sample efficiency for estimating the WPFD scale parameter, while maintaining asymptotic comparability with the maximum likelihood approach.

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INTRODUCTION

Statistical inference is primarily guided by two major approaches: the classical (or frequentist) and the Bayesian methods. The classical approach, developed by R.A. Fisher in the 1930s, treats parameters as fixed but unknown quantities. In contrast, the Bayesian method, named after Thomas Bayes, considers parameters as random variables with uncertain values. The fundamental distinction lies in this treatment of parameters: classical inference assumes fixed parameters, while Bayesian inference models them as probabilistic. Despite the dominance of frequentist methods in practice, Bayesian approaches have gained significant traction and are now standard in many modern applications.

In Bayesian statistics, a statistical model is constructed to link observed data with underlying parameters, incorporating prior information about those parameters. Before data collection, prior distributions represent initial beliefs about the parameters. Once data are available, these beliefs are updated using the likelihood function to yield the posterior distribution, which reflects a combination of prior knowledge and data-based evidence.

Tahir et al. (2016) introduced the four-parameter Weibull-Power Function Distribution (WPFD), an extension of the traditional Power Function Distribution. Their study demonstrated the distribution's desirable properties and practical applicability to real-world datasets. The WPFD was shown to be more flexible than related distributions. Similarly, several other models have been proposed and proven useful in diverse fields such as medicine, engineering, survival analysis, insurance, hydrology, and economics. Examples include the works of Adepoju et al. (2024a, 2024b), Isa et al. (2023), Kajuru et al. (2023), Adepoju et al. (2023), Bello et al. (2020, 2021), and Ibrahim et al. (2020a, 2020b), among others.

To estimate parameters of newly developed distributions, researchers have employed various classical estimation techniques, as seen in studies such as Adepoju et al. (2024c, 2024d), Hassan et al. (2023), Yilmaz et al. (2021), and ZeinEldin et al. (2019). Classical estimation methods do not require prior information about parameters. In contrast, Bayesian estimation relies on selecting appropriate prior distributions.

Correspondence: Ibrahim Abubakar Sadiq. Department of Statistics, Faculty of Science, Ahmadu Bello University, Zaria, Nigeria. ✉ isabubakar@abu.edu.ng.

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Danrimi and Abubakar (2023) applied a Bayesian approach to estimate parameters of the two-parameter Weibull distribution, using a gamma prior. Their analysis showed that Bayesian estimates outperformed those obtained via the maximum likelihood method. Similarly, Liu et al. (2021) compared classical and Bayesian methods for estimating parameters of the Power Function Distribution. Using conjugate priors under five loss functions, Squared Error, Precautionary, Weighted, DeGroot, and Linex, they found that Bayesian estimates were more efficient than MLEs.

Further comparative studies, such as that by Adepoju et al. (2021a), examined Bayesian estimation using extended Jeffreys and quasi priors under three loss functions versus the classical method. Their findings favoured the Bayesian approach, particularly under the quadratic loss function. Adepoju et al. (2021b) evaluated the performance of Bayesian and classical methods for estimating the scale parameter of the Inverse Rayleigh-Frechet distribution, concluding that the quadratic loss function yielded superior results across various priors.

Several other researchers have also reported favourable outcomes for Bayesian methods over MLEs, especially when evaluated using Monte Carlo simulations. Notable contributions include those by Eraikhuemen et al. (2020a, 2020b), Ieren et al. (2020), Ieren and Oguntunde (2018), Preda et al. (2010), Dey (2010), and Aliyu and Abubakar (2016).

Some other distributions were developed and found to be powerful, making them a more useful candidate in various fields such as medical, engineering, survival analysis, insurance, hydrology, economics, and so on. Such a model can be found in Sadiq *et al* (2022), Sadiq *et al* (2024), Sadiq *et al* (2023a), Kajuru *et al* (2023), Sadiq *et al* (2023b), Mohammed *et al.* (2025), Sadiq *et al* (2023c), Obafemi *et al* (2024), Habu *et al* (2024), Semary *et al* (2025), Sadiq *et al.* (2025a) and Abd Elgawad *et al.* (2025), Sadiq *et al.* (2025b), Mohammed *et al.* (2025), Dangana *et al.* (2025), Oga *et al.* (2025), Usman *et al.* (2025), Yusuf *et al.* (2025), Sadiq *et al.* (2026) to mention but few.

Despite the growing body of literature on Bayesian estimation for lifetime distributions, several gaps remain in the context of the Weibull-Power Function Distribution (WPFD). Existing studies on WPFD have primarily focused on distributional properties, model flexibility, and classical estimation techniques, with limited attention given to Bayesian inference (Dangana *et al.*, 2025). Moreover, where Bayesian methods have been applied to related Weibull-type or power-function families, the emphasis has often been on shape parameters or on a single loss function, with little comparative analysis across different loss structures or prior specifications.

The present study fills this gap by providing a focused Bayesian analysis of the scale parameter of the WPFD, which plays a critical role in reliability and survival applications. Unlike previous works, this paper derives closed-form Bayes estimators of the scale parameter under three distinct loss functions, squared error, quadratic, and precautionary, thereby allowing a systematic assessment of the impact of loss-function choice on estimator performance. In addition, the study conducts a comprehensive comparison between an informative Gamma prior and two commonly used non-informative priors (uniform and Jeffreys), an aspect that has not been jointly examined for the WPFD in existing literature.

To the best of our knowledge, no prior study has simultaneously investigated Bayesian estimation of the WPFD scale parameter under multiple loss functions and contrasting prior assumptions, nor compared these estimators directly with the maximum likelihood estimator using extensive Monte Carlo simulations. By addressing this gap, the present work provides practical guidance on prior and loss-function selection for WPFD scale estimation and contributes to the broader literature on Bayesian inference for flexible lifetime distributions. However, Bayesian estimation of the WPFD scale parameter under alternative loss functions has not yet been explored.

NOTATIONS

For ease of reference and to enhance reproducibility of the derivations, we summarise the principal notation used throughout the manuscript in Table 1.

Estimation Methods

Tahir et al. (2016) classify the probability density function (PDF) and cumulative distribution function (CDF) of the Weibull-Power Function Distribution (WPFDt) as follows:

$$f_{WPFDt}(z) = \frac{pnv^q q z^{qn-1}}{(v^q - z^v)^{n+1}} e^{-p\left[\frac{z^v}{v^q - z^v}\right]^n} \tag{1}$$

$$F_{WPFDt}(z) = 1 - e^{-p\left[\frac{z^v}{v^q - z^v}\right]^n} \tag{2}$$

For z>0 and v, p, n, q>0, p and v are shape parameters, while q and n are scale parameters.

Maximum Likelihood Estimation Method

When a set of random observations, denoted as$z_1, z_2, \dots, z_n$, are drawn from a population Z characterised by a$f_{WPFDt}(z)$, the likelihood function,$B(Z|p,n,v,q)$, represents the combined probability density of these individual observations. Furthermore, the specific mathematical formula for the probability density pdf of the WPFDt is presented in equation (1).

The chances function is specified by;

$$B(Z|p,n,v,q) \propto (pnv^q q)^w \prod_{i=1}^{w} \left(\frac{z^{qn-1}}{(v^q - z^q)^{n+1}}\right) e^{-p\sum_{i=1}^{n}\left[\frac{z^q}{v^q - z^q}\right]^n} \tag{3}$$

Table 1: Summary of the principal notation used throughout the Research.

Symbol	Description
Z	Random variable following the Weibull–Power Function Distribution (WPFD)
$z_1, z_2, \dots, z_n$	Random sample of size n from WPFD
n	Sample size
p	Scale parameter of the WPFD (parameter of interest)
v, q	Shape parameters of the WPFD
η	Auxiliary transformation variable used in integration
$f(z; p)$	Probability density function of WPFD
$F(z; p)$	Cumulative distribution function of WPFD
$L(p)$	Likelihood function
$\ell(p)$	Log-likelihood function
$\pi(p)$	Prior distribution of the parameter p
$\pi_J(p)$	Jeffreys prior
a, b	Hyperparameters of the Gamma informative prior
$g(p \mid z)$	Posterior distribution of p
$I(p)$	Fisher information for the parameter p
SELF	Squared Error Loss Function
QLF	Quadratic Loss Function
PLF	Precautionary Loss Function
MSE	Mean Squared Error
$\hat{p}_{MLE}$	Maximum Likelihood Estimator of p
$\hat{p}_B$	Bayesian estimator of (p)

The chances function for p is prearranged by;

$$B(z|p) = \eta p^{w} e^{-p\sum_{i=1}^{n}\left[\frac{z^q}{v^q - z^q}\right]^n} \tag{4}$$

Where $\eta = (nv^q q)z_i^{qn-1}(v^q - z_i^q)^{-n-1}$ is a constant which is independent of the shape parameter$p$.

By partially differentiating $B$with respect to $p$and solving for$\hat{p}$,

$$\frac{\partial B}{\partial p} = \frac{w}{p} - \sum_{i=1}^{w}\left[\frac{z^q}{v^q - z^q}\right]^n \tag{5}$$

$$\Rightarrow \hat{p} = w\left(\sum_{i=1}^{w}\left[\frac{z^q}{v^q - z^q}\right]^n\right)^{-1} \tag{6}$$

Combining the likelihood function with the Gamma informative prior using Bayes' theorem yields the posterior distribution of the scale parameter, which also follows a Gamma-type distribution with updated hyperparameters. This posterior distribution is subsequently used to derive Bayes estimators under the squared error, quadratic, and precautionary loss functions (Dangana *et al.*, 2025).

BAYESIAN ESTIMATION METHOD

Posterior Distributions

The likelihood function represents the joint probability distribution of the observed data. However, it is conceptualised as a function of the model's parameters, with the collected data points considered as fixed values. Given that the data points,$\underline{z} = (z_1, z_2, \dots, z_n)$, are independently obtained, the likelihood function can then be formulated.

$$B(\underline{z}|p, n, v, q) = B(z_1, z_2, \dots, z_n|p, n, v, q) = \prod_{i=1}^{w} B(\underline{z}|p, n, v, q) \tag{7}$$

It is important to note that, for a given$z$, the likelihood is expressed as a function of p, whereas for a given p, the pdf is expressed as a function of z.

To determine the posterior distribution, denoted as $B(p|\underline{z})$, which illustrates the probability distribution of a parameter once the relevant data has been observed, we employ Bayes' theorem.

$$b(p|\underline{z}) = \frac{b(p)B(\underline{z}|p)}{g(z)} \tag{8}$$

Where g(z) denotes the marginal distribution of Z and

$$g(z) = \int_{-\infty}^{\infty} b(p)B(\underline{z}|p) \tag{9}$$

wherever $b(p)$ represents the prior distribution and $B(\underline{z}|p)$ represents the likelihood function.

Posterior Distribution of the Scale Parameter under the Assumption of Uniform Prior

The uniform prior, when applied to the shape parameter$p$serves as a non-informative prior. This means it assigns equal probability to all possible values of$p$, thereby reflecting a lack of strong prior beliefs or information about the parameter.

$$b(p) \propto 1; 0 < p < \infty \tag{10}$$

Recall that the posterior distribution of the shape parameter λ is defined as:

$$b(p|\underline{z}) = \frac{b(p)B(p|\underline{z})}{\int_0^{\infty} b(p)B(p|\underline{z})dp} \tag{11}$$

It is important to remember that the likelihood function for the WPFDt, specifically concerning its scale parameter, is provided by a particular formulation of that equation.

$$B(Z|p,n,v,q) \propto$$

$$(pnv^q q)^w \prod_{i=1}^{w} \left(\frac{z^{qn-1}}{(v^q-z^q)^{n+1}}\right) e^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} \tag{12}$$

and

$$B(Z|p) \propto p^w e^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} \tag{13}$$

Now, let

$$H = \int B(\underline{z}|p)b(p)dp \tag{14}$$

Substituting for $b(p)$ and$B(\underline{z}|p)$, we encompass:

$$H = \eta \int_0^\infty p^w \mathrm{e}^{-p\sum_{i=1}^{n}(\mathrm{e}^{(qz_i^{-1})^n}-1)^{-1}} dp \tag{15}$$

Moreover, by applying the integration by substitution method to equation (15), the subsequent result is derived:

Let

$$u = p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n \Rightarrow p = \frac{u}{\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} \tag{16}$$

$$dp = \frac{du}{\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} \tag{17}$$

By substituting the expressions for $p$and dpinto equation (15) and subsequently simplifying the resulting equation, we obtain the following expression.

$$H = \eta \int_0^\infty \left(\frac{u}{\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}\right)^w \mathrm{e}^{-u} \frac{du}{\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} \tag{18}$$

$$H = \eta \frac{1}{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}} \int_0^\infty u^w \mathrm{e}^{-u} du \tag{19}$$

As well as evoke that $\int_{-\infty}^{\infty} y^{t-1}e^{-y}dy = \Gamma(t)$along with that $\int_{-\infty}^{\infty} y^t e^{-y}dy = \Gamma(t+1)$

Consequently;

$$H = \frac{\eta\Gamma(t+1)}{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}} \tag{20}$$

The posterior distribution under a uniform prior is obtained by substituting for$H$, $b(p)$and $B(\underline{z}|p)$in equation (13) and simplifying. The resulting expression is as follows:

$$B(p|\underline{z}) = \frac{\eta p^w \mathrm{e}^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}}{\frac{\eta\Gamma(t+1)}{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}}} \tag{21}$$

$$B(p|\underline{z}) = \frac{p^w\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}}{\Gamma(t+1)\mathrm{e}^{p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}} \tag{22}$$

Posterior Distribution of the Scale Parameter under the Assumption of Jeffrey's Prior

Jeffreys' Prior for the Scale Parameter

Jeffreys' prior for a parameter p is defined by (Bernardo *et al.*, 1994) as

$$\pi_J(p) \propto \sqrt{I(p)} \tag{23}$$

where $I(p)$ denotes the Fisher information given by:

$$I(p) = -\mathbb{E}\left[\frac{\partial^2}{\partial p^2} log\, f\,(Z;p)\right] \tag{24}$$

For the Weibull-Power Function Distribution, the scale parameter p enters the probability density function through the ratio z/p, indicating that p is a pure scale parameter. For distributions of this form, the Fisher information satisfies (Dangana *et al.*, 2025):

$$I(p) \propto \frac{1}{p^2} \tag{25}$$

a result that holds generally for scale families under regularity conditions.

Consequently, the Jeffreys prior for the WPFD scale parameter is obtained (Dangana *et al.*, 2025) as:

$$\pi_J(p) \propto \sqrt{I(p)} \propto \frac{1}{p}, \quad p > 0 \tag{26}$$

This choice ensures invariance under reparameterization and has been widely adopted for scale parameters in Bayesian reliability and lifetime modelling (Kass and Wasserman, 1996).

The Jeffrey's non-informative prior for the WPFDt shape parameter p is defined (Dangana *et al.*, 2025) as:

$$b(p) \propto \frac{1}{p}; 0 < p < \infty \tag{27}$$

The posterior distribution of the shape parameter p, given the data and using Jeffrey's prior, is defined (Dangana *et al.*, 2025) as:

$$b(p|\underline{z}) = \frac{b(p)B(p|\underline{z})}{\int_0^\infty b(p)B(p|\underline{z})dp} \tag{28}$$

Now, let

$$H = \int B(\underline{z}|p)B(p)dp \tag{29}$$

Putting for $b(p)$ and$B(\underline{z}|p)$; we encompass:

$$H = \eta \int_0^\infty p^{w-1}\mathrm{e}^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} d\alpha \tag{30}$$

Furthermore, by applying the routine of integration by substitution to equation (30) and simplifying, we acquire:

$$H = \eta \int_0^{\infty} \left(\frac{u}{\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n}\right)^{w-1} \mathrm{e}^{-u} \frac{du}{\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n} \quad (31)$$

$$H = \eta \frac{1}{[\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n]^w} \int_0^{\infty} u^{w-1} \mathrm{e}^{-u} du \quad (32)$$

$$H = \frac{\eta \Gamma(w)}{\left[\sum_{i=1}^{w} \left[\frac{z^q}{v^q - z^q}\right]^n\right]^w} \quad (33)$$

By substituting the expressions for$H$, $b(p)$ and $B(\underline{z}|p)$ in into equation (28) and then simplifying the result, we derive the posterior distribution for the parameter under the assumption of a Jeffrey's prior. The resulting expression is as follows:

$$B(p|\underline{z}) = \frac{[\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n]^w p^{w-1} \mathrm{e}^{-p \sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n}}{\Gamma(w)} \quad (34)$$

Informative Prior Specification

In addition to the non-informative uniform and Jeffreys' priors, an informative prior distribution is assumed for the scale parameter of the Weibull-Power Function Distribution. Specifically, the scale parameter θ is assumed to follow a Gamma distribution with shape parameter $a > 0$ and rate parameter $a > 0$ (Dangana *et al.*, 2025), denoted by:

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} exp\{-b\theta\}, \quad \theta > 0 \quad (35)$$

The Gamma prior is chosen due to its flexibility, support on the positive real line, and its widespread use as an informative prior for scale parameters in reliability and survival analysis. Moreover, it facilitates analytical tractability in Bayesian inference (Dangana *et al.*, 2025).

The hyperparameters a and b are selected to reflect moderate prior information about the scale parameter. In particular, the prior mean $E(\theta) = a/b$ is set close to the true scale parameter value used in the simulation study, while the prior variance $\mathrm{Var}(\theta) = a/b^2$ is chosen to allow reasonable dispersion, thereby avoiding excessive prior dominance (Dangana *et al.*, 2025). In the simulation study, we set $a = 1$ and $b = 1$, corresponding to a prior mean of 0.5 and a variance of 0.125.

Bayesian Estimation under Uniform Prior Using Three Loss Functions

We estimate the WPFDt's scale parameter using the posterior distribution derived from the uniform prior. This estimation is evaluated against three different loss functions, where a loss function,$B(p, p_{PLFt})$, defines the cost of an estimate, δ, deviating from the true parameter value, $\hat{n}$.

Under Squared Error Loss Function (SELF$_t$)

The squared error loss function, which we will use to estimate the parameter p, is defined as follows

$$B(p, p_{SELFt}) = (p - p_{SELFt})^2 \quad (36)$$

The Bayes estimator under a uniform prior and Squared Error Loss Function (SELFt) is consequent as:

$$p_{SELFt} = E(p|\underline{z}) \quad (37)$$

$$E(p|\underline{z}) = \int_0^{\infty} pb(p|\underline{z})\, dp \quad (38)$$

$$B(p|\underline{z}) = \frac{p^n[\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n]^{w+1}}{\Gamma(w+1)\mathrm{e}^{p \sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n}} \quad (39)$$

Putting for $B(p|\underline{z})$ in equation (39), we comprise:

$$E(p|\underline{z}) = \frac{[\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n]^{w+1}}{\Gamma(w+1)} \int_0^{\infty} p^{w+1} \mathrm{e}^{-p \sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n} dp \quad (40)$$

We first apply the method of integration by substitution to equation (40). After subsequent simplification, the expression becomes:

$$E(p|\underline{z}) = (w+1) \left[\sum_{i=1}^{w} \left[\frac{z^q}{v^q - z^q}\right]^n\right]^{-1} \quad (41)$$

Under Quadratic Loss Function (QLFt)

The quadratic loss function is distinct (Dangana *et al.*, 2025; Bernardo *et al.*, 1994; Kass and Wasserman, 1996) as:

$$B(p, p_{QLFt}) = \left(\frac{p - p_{QLFt}}{p}\right)^2 \quad (42)$$

The Bayes estimator under a uniform prior and the Quadratic Loss Function (QLFt) is derived as:

$$p_{QLFt} = \frac{E(p^{-1}|\underline{z})}{E(p^{-2}|\underline{z})} = \frac{\int_0^{\infty} p^{-1} B(p|\underline{z}) dp}{\int_0^{\infty} p^{-2} B(p|\underline{z}) dp} \quad (43)$$

$$E(p^{-1}|\underline{z}) = \int_0^{\infty} p^{-1} B(p|\underline{z}) dp \quad (44)$$

Now, recall that under the assumption of a uniform prior,

$$B(p|\underline{z}) = \frac{p^n[\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n]^{w+1}}{\Gamma(w+1)\mathrm{e}^{p \sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n}} \quad (45)$$

By substituting the value of $B(p|\underline{z})$ into equation (45), we obtain:

$$E(p^{-1}|\underline{z}) = \frac{[\sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n]^{w+1}}{\Gamma(w+1)} \int_0^{\infty} p^{w-1} \mathrm{e}^{-p \sum_{i=1}^{w} [\frac{z^q}{v^q - z^q}]^n} dp \quad (46)$$

We first apply the method of integration by substitution to the expression in equation (46). After subsequent simplification, the result is:

$$E\left(p^{-1}|\underline{z}\right)=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]\Gamma(w)}{\Gamma(w+1)} \tag{47}$$

$$E\left(p^{-2}|\underline{z}\right)=\int_0^\infty p^{-2}B\left(p|\underline{z}\right)dp \tag{48}$$

It should be recalled that, under the assumption of a uniform prior distribution,

$$B(p|\underline{z})=\frac{p^w\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}}{\Gamma(w+1)\mathrm{e}^{p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}} \tag{49}$$

Upon substituting the value of $B\left(p|\underline{z}\right)$into equation (48), we obtain:

$$E(p^{-2}|\underline{z})=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}}{\Gamma(w+1)}\int_0^\infty p^{n-2}\mathrm{e}^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}dp \tag{50}$$

By applying the method of integration by substitution to equation (50) and simplifying, we obtain:

$$E\left(p^{-2}|\underline{z}\right)=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^2\Gamma(w-1)}{\Gamma(w+1)} \tag{51}$$

$$p_{QLFt}=\frac{E(p^{-1}|\underline{z})}{E(p^{-2}|\underline{z})} \tag{52}$$

This implies that

$$p_{QLFt}=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]\Gamma(w)}{\Gamma(w+1)}\div\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^2\Gamma(w-1)}{\Gamma(w+1)} \tag{53}$$

$$p_{QLFt}=\frac{(w-1)}{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]} \tag{54}$$

Using Precautionary Loss Function (PLFt)

The precautionary loss function (*PLFt*) as

$$b(p_{PLFt},p)=\frac{(p_{PLFt}-p)^2}{p} \tag{55}$$

Similarly, the derivation of the Bayes estimator using PLFt under a uniform prior is obtained as follows:

$$p_{PLF}=\{E(p^2|\underline{z})\}^{\frac{1}{2}}=\sqrt{E(p^2|\underline{z})} \tag{56}$$

$$E(p^2|\underline{z})=\int_0^\infty p^2B(p|\underline{z})dp \tag{57}$$

Recall that under the assumption of a uniform prior,

$$B(p|\underline{z})=\frac{p^w\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}}{\Gamma(w+1)\mathrm{e}^{p\sum_{i=1}^{w}\left(\mathrm{e}^{(qz_i^{-1})^n}-1\right)^{-1}}} \tag{58}$$

By substituting$B(p|\underline{z})$ into equation (57), we obtain:

$$E(p^2|\underline{z})=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{w+1}}{\Gamma(w+1)}\int_0^\infty p^{n+2}\mathrm{e}^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}dp \tag{59}$$

Applying integration by substitution to equation (59) and simplifying, we obtain:

$$E(p^2|\underline{z})=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-2}}{\Gamma(w+1)}\int_0^\infty u^{w+3-1}\mathrm{e}^{-u}du \tag{60}$$

$$E\left(p^2|\underline{z}\right)=\frac{\Gamma(w+3)\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-2}}{\Gamma(w+1)} \tag{61}$$

$$\lambda_{PLF}=\left\{E\left(p|\underline{z}\right)\right\}^{\frac{1}{2}}=\left\{(w+1)(w+2)\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-2}\right\}^{\frac{1}{2}} \tag{62}$$

$$p_{PLFt}=[(w+1)(w+2)]^{\frac{1}{2}}\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-1} \tag{63}$$

Bayesian Estimation under Jeffrey's Prior Using Three Loss Functions

The scale parameter of the WPFDt is estimated under three loss functions, based on the posterior distribution derived from Jeffrey's prior.

Using Squared Error Loss Function (SELFt)

The Bayes estimator under SELFt with Jeffrey's prior is derived as:

$$p_{SELFt}=E(p|\underline{z}) \tag{64}$$

$$E(p|\underline{z})=\int_0^\infty pB(p|\underline{z})\,dp \tag{65}$$

Now recall that for Jeffrey's prior,

$$B(p|\underline{z})=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^w p^{w-1}\mathrm{e}^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}}{\Gamma(w)} \tag{66}$$

By substituting $B\left(p|\underline{z}\right)$into equation (65), we obtain:

$$E(p|\underline{z})=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^w}{\Gamma(w)}\int_0^\infty p^w\mathrm{e}^{-p\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n}dp \tag{67}$$

Applying integration by substitution to equation (67) and simplifying, we obtain:

$$E(p|\underline{z})=\frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-1}}{\Gamma(w)}\int_0^\infty u^{w+1-1}\mathrm{e}^{-u}du \tag{68}$$

$$p_{SELFt}=E\left(p|\underline{z}\right)=w\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-1} \tag{69}$$

Under Quadratic Loss Function (QLFt)

The Bayes estimator under QLFt with Jeffrey's prior is derived as:

$$p_{QLFt} = \frac{E(p^{-1}|\underline{z})}{E(p^{-2}|\underline{z})} = \frac{\int_0^\infty p^{-1}B(p|\underline{z})dp}{\int_0^\infty p^{-2}B(p|\underline{z})dp} \quad (70)$$

$$E\left(p^{-1}|\underline{z}\right) = \int_0^\infty p^{-1}B\left(p|\underline{z}\right)dp \quad (71)$$

At this moment, remember that for Jeffrey's prior,

$$B(p|\underset{-}{z}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^w p^{w-1}\mathrm{e}^{-p\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n}}{\Gamma(w)} \quad (72)$$

Substituting for $B(p|\underline{z})$ in equation (71), we have:

$$E(p^{-1}|\underset{-}{z}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^w}{\Gamma(w)}\int_0^\infty p^{w-2}\mathrm{e}^{-p\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n}dp \quad (73)$$

Applying integration by substitution to equation (73) and simplifying, we obtain:

$$E\left(w^{-1}|\underline{z}\right) = \frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]}{(w-1)} \quad (74)$$

$$E\left(p^{-2}|\underline{z}\right) = \int_0^\infty p^{-2}B\left(p|\underline{z}\right)dp \quad (75)$$

Now recall that for Jeffrey's prior,

$$B(p|\underset{-}{z}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^w p^{w-1}\mathrm{e}^{-p\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n}}{\Gamma(w)} \quad (76)$$

Substituting for $B(p|\underline{z})$ in equation (75), we encompass:

$$E(p^{-2}|\underset{-}{z}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^w}{\Gamma(w)}\int_0^\infty p^{n-3}\mathrm{e}^{-p\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n}dp \quad (77)$$

On applying the substitution method to equation (77) and simplifying, the result is

$$E\left(p^{-2}|\underline{z}\right) = \frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^2}{(w-1)(w-2)} \quad (78)$$

But recollect that

$$p_{QLFt} = \frac{E(p^{-1}|\underline{z})}{E(p^{-2}|\underline{z})} \quad (79)$$

This implies that

$$p_{QLFt} = \frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]}{(w-1)} \div \frac{\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^2}{(w-1)(w-2)} \quad (80)$$

$$p_{QLFt} = \frac{(w-2)}{\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n} \quad (81)$$

Using Precautionary Loss Function (PLFt)

Similarly, the derivation of the Bayes estimator under PLFt with Jeffrey's prior is obtained, following the methodology of Azam and Ahmad (2014):

$$p_{PLFt} = \{E(p^2|\underline{z})\}^{\frac{1}{2}} = \sqrt{E(p^2|\underline{z})} \quad (82)$$

$$E(p^2|\underline{z}) = \int_0^\infty p^2B(p|\underline{z})dp \quad (83)$$

As a reminder, for Jeffrey's prior

$$B(p|\underset{-}{z}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^w p^{w-1}\mathrm{e}^{-p\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n}}{\Gamma(w)} \quad (84)$$

Upon substituting for $B(p|\underline{z})$ in equation (83), the following expression is obtained:

$$E(p^2|\underset{-}{w}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^w}{\Gamma(w)}\int_0^\infty p^{w+1}\mathrm{e}^{-p\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n}dp \quad (85)$$

The expression in equation (85) is solved using integration by substitution and simplified, resulting in:

$$E(p^2|\underset{-}{z}) = \frac{[\sum_{i=1}^{w}[\frac{z^q}{v^q-z^q}]^n]^{-2}}{\Gamma(w)}\int_0^\infty u^{w+2-1}\mathrm{e}^{-u}du \quad (86)$$

$$E\left(p^2|\underline{z}\right) = w(w+1)\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-2} \quad (87)$$

$$p_{PLFt} = [w(w+1)]^{\frac{1}{2}}\left[\sum_{i=1}^{w}\left[\frac{z^q}{v^q-z^q}\right]^n\right]^{-1} \quad (88)$$

DETAILED DERIVATION FOR THE INFORMATIVE PRIOR UNDER THE QUADRATIC LOSS FUNCTION

To enhance transparency, we present one representative derivation in full detail: the posterior distribution and Bayes estimator under the informative Gamma prior and Quadratic Loss Function (QLF) (Dangana *et al.*, 2025).

Step 1: Likelihood Function

Let $z_1, z_2, \dots, z_n$ be a random sample from the WPFD with scale parameter p. The likelihood function (Dangana *et al.*, 2025) is:

$$L(p) = \prod_{i=1}^{n} f\,(z_i; p) \quad (89)$$

For the WPFD, the likelihood can be written in the general scale-family form:

$$L(p) \propto p^{-n}\,exp\left(-\frac{S}{p}\right) \quad (90)$$

where S is a sufficient statistic depending on the sample.

Step 2: Informative Gamma Prior

Assuming the informative prior is given as:

$$\pi(p) = \frac{b^a}{\Gamma(a)} p^{a-1} e^{-bp}, \quad p > 0 \tag{91}$$

where $a > 0, b > 0$.

Step 3: Posterior Distribution

By Bayes' theorem,

$$g(p|z) \propto L(p)\pi(p) \tag{92}$$

Substituting,

$$g(p|z) \propto p^{-n} e^{-S/p} \cdot p^{a-1} e^{-bp} \tag{93}$$

Rearranging powers of p:

$$g(p|z) \propto p^{a-n-1} \, exp\left(-bp - \frac{S}{p}\right) \tag{94}$$

Step 4: Normalising Constant

The posterior normalising constant is:

$$C^{-1} = \int_0^\infty p^{a-n-1} \, exp\left(-bp - \frac{S}{p}\right) dp \tag{95}$$

This integral converges provided: $a - n > 0; b > 0; S > 0$

The integral is finite under these conditions due to exponential decay at both 0 and ∞ (Dangana *et al.*, 2025).

Step 5: Bayes Estimator under Quadratic Loss Function

Under QLF, the Bayes estimator is

$$\hat{p}_{QLF} = \frac{E(p^{-1}|z)}{E(p^{-2}|z)} \tag{96}$$

Using the posterior distribution,

$$E(p^{-k}|z) = \int_0^\infty p^{-k} \, g(p|z) dp \tag{97}$$

Substituting the posterior kernel,

$$E(p^{-k}|z) \propto \int_0^\infty p^{a-n-k-1} \, exp\left(-bp - \frac{S}{p}\right) dp \tag{98}$$

provided: $a - n - k > 0$the integral converges. Thus, the QLF estimator is obtained explicitly as a ratio of finite posterior moments.

Computational and Algorithmic Implementation

The posterior distributions derived under the Gamma informative prior and the two non-informative priors (uniform and Jeffreys) belong to standard Gamma-type families. Consequently, the posterior expectations required for the Bayes estimators under the squared error loss function (SELF), quadratic loss function (QLF), and precautionary loss function (PLF) admit closed-form expressions.

Specifically:

i. Under **SELF**, the Bayes estimator corresponds to the posterior mean, which is available in closed form.

ii. Under **QLF**, the estimator reduces to a ratio of posterior expectations, which also simplifies analytically.

iii. Under **PLF**, the estimator involves expectations of inverse powers of the parameter, which are available in closed form for Gamma-type posteriors.

Therefore, no numerical integration was required in deriving the Bayes estimators presented in this study.

For the Monte Carlo simulation study, data generation and estimator evaluation were implemented in R (version 4.4.3). Each simulation scenario was replicated 1000 times. Random variates from the WPFD were generated using the inverse transform method. Convergence of simulation summaries was verified by increasing the number of replications and confirming the stability of the estimated mean squared errors to three decimal places.

SIMULATION RESULTS

Monte Carlo Design

To assess the finite-sample performance of the proposed estimators, a Monte Carlo simulation study was conducted. For each parameter configuration and sample size, **R = 1000 independent replications** were generated. All simulations were implemented in **R version 4.4.3**. Random samples from the Weibull-Power Function Distribution (WPFD) were generated using the inverse transform method. A fixed random seed was set at the beginning of each simulation scenario to ensure reproducibility (Table 2).

For each replication:

i. The Maximum Likelihood Estimator (MLE) of the scale parameter p was obtained via numerical maximisation of the log-likelihood function.

ii. Bayesian estimators under the Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), and Precautionary Loss Function (PLF) were computed from the corresponding posterior distributions.

iii. When closed-form expressions were not available, numerical integration was performed using R's built-in integrate() function with default adaptive quadrature and tolerance settings.

The performance of each estimator was evaluated using:

The Bias was computed as:

$$\text{Bias} = \frac{1}{R}\sum_{r=1}^{R}(\hat{p}_r - p) \quad (99)$$

The Mean Squared Error (MSE) was computed as:

$$\text{MSE} = \frac{1}{R}\sum_{r=1}^{R}(\hat{p}_r - p)^2 \quad (100)$$

and equivalently verified as:

$$\text{MSE} = (\text{Bias})^2 + \text{Var}(\hat{p}) \quad (101)$$

The Monte Carlo Standard Deviation (MC SD) was computed as:

$$\text{MC SD} = \sqrt{\frac{1}{R-1}\sum_{r=1}^{R}(\hat{p}_r - \bar{p})^2} \quad (102)$$

The Root Mean Squared Error (RMSE) was computed as:

$$\text{RMSE} = \sqrt{\text{MSE}} \quad (103)$$

The Monte Carlo Standard Error of the Mean (MC SE) was computed as:

$$\text{MCSE}(\bar{p}) = \frac{\text{MC SD}}{\sqrt{R}} \quad (104)$$

where (R = 1000). Although 1000 replications provide stable performance estimates, increasing the number of replications may further reduce Monte Carlo error.

Table 2: Monte Carlo Simulation Design

Item	Value
Number of replications	1000
Software	R version 4.4.3
True parameter values	p=0.5
Fixed parameters	η=0.5,ν=2.5,q=0.5
Prior (Uniform)	Beta(1,1)
Prior (Jeffrey)	Beta(0.5,0.5)
Loss Functions	SELF, QLF, PLF
Sample sizes	25, 50, 100, 200, 300, 400, 500

Table 3 reports the Monte Carlo performance of the MLE and Bayesian estimators of the scale parameter p (true value 0.50) based on 1000 replications. For small samples (n = 25), all estimators exhibit negative bias, with the MLE (-0.0569) and U-SELF (-0.0610) showing the largest underestimation, while the QLF-based estimators, particularly J-QLF (-0.0223), demonstrate noticeably smaller bias. Across all sample sizes, the QLF estimators consistently produce the lowest MSE values, with J-QLF achieving the best overall performance (e.g., MSE = 0.0697 at n = 25 and 0.0590 at n = 500). Correspondingly, J-QLF also records the smallest RMSE values, indicating improved estimation precision relative to both the MLE and other Bayesian loss functions. As the sample size increases from 25 to 500, the bias steadily moves toward zero for most estimators, confirming consistency. The coverage probabilities improve monotonically, approaching or slightly exceeding the nominal 95% level, with J-QLF again providing the highest coverage (0.978 at n = 500). Although the MLE remains competitive, particularly for moderate and large samples, the Bayesian estimators under quadratic loss, especially with Jeffreys prior, demonstrate superior overall performance in terms of bias reduction, MSE, and interval coverage. These results suggest that the QLF-based Bayesian approach offers improved small- and moderate-sample efficiency for estimating the scale parameter p.

Table 4 presents the Monte Carlo performance of the estimators of the scale parameter p (true value 0.50) under the alternative configuration ($\eta = 2.5, \nu = 0.5, q = 0.5$), based on 1000 replications. For small samples (n = 25), all estimators exhibit negative bias, with MLE (-0.0567) and U-SELF (-0.0610) showing the largest underestimation, while the QLF-based estimators again display noticeably smaller bias (e.g., J-QLF = -0.0229). Across all sample sizes, the quadratic loss estimators consistently achieve the lowest MSE values; in particular, J-QLF attains the minimum MSE throughout (e.g., 0.0697 at n = 25 and 0.0544 at n = 500), accompanied by the smallest RMSE values. The PLF estimators are generally competitive with MLE and SELF but do not outperform the QLF approach in terms of overall accuracy. As the sample size increases from 25 to 500, the bias steadily diminishes toward zero for all methods, confirming consistency. Coverage probabilities improve with increasing n, approaching and slightly exceeding the nominal 95% level, with J-QLF again providing the highest coverage (0.979 at n = 500). The general pattern closely mirrors that observed in Table 3, indicating that the change in η does not materially alter the relative ranking of estimators. In summary, the Bayesian estimator under quadratic loss with Jeffreys prior demonstrates the most favourable performance in terms of bias reduction, MSE minimisation, and interval coverage across small, moderate, and large sample sizes.

Table 5 reports the Monte Carlo performance of the estimators of the scale parameter p (true value 0.50) under the configuration ($\eta = 0.5, \nu = 2.5, q = 0.5$), based on 1,000 replications. Unlike the previous settings, the variance and MSE decrease sharply as the sample size increases, clearly reflecting the expected 1/n convergence behaviour. For n = 25, all estimators exhibit moderate negative bias (around -0.05 for MLE and SELF-type estimators), while the quadratic loss estimators show smaller bias (approximately -0.02). The J-QLF estimator achieves the smallest MSE (0.0102) and RMSE (0.101) at n = 25, indicating improved efficiency in small samples.

Table 3: Performance of Estimators for p with Fixed Parameters ($p = 0.5$, $\eta = 0.5$, $\nu = 0.5$, $q = 0.5$, $a = 1.0$, $b = 1$)

n	Measure	MLE	U-SELF	U-QLF	U-PLF	J-SELF	J-QLF	J-PLF
25	Estimate	0.4431	0.439	0.477	0.4479	0.4444	0.4777	0.448
25	Bias	-0.0569	-0.061	-0.023	-0.0521	-0.0556	-0.0223	-0.052
25	Variance	0.0711	0.0691	0.0707	0.071	0.0683	0.0692	0.075
25	MSE	0.0743	0.0728	0.0712	0.0737	0.0714	0.0697	0.0777
25	RMSE	0.2726	0.2698	0.2668	0.2715	0.2672	0.264	0.2787
25	SE(MSE)	0.0021	0.0019	0.0018	0.002	0.0018	0.0017	0.0022
25	95% Cov	0.912	0.934	0.941	0.928	0.945	0.952	0.919
50	Estimate	0.444	0.4369	0.4771	0.4487	0.4453	0.4875	0.4487
50	Bias	-0.056	-0.0631	-0.0229	-0.0513	-0.0547	-0.0125	-0.0513
50	Variance	0.07	0.0687	0.0704	0.0704	0.0681	0.0621	0.0745
50	MSE	0.0731	0.0727	0.0709	0.073	0.0711	0.0623	0.0771
50	RMSE	0.2704	0.2696	0.2663	0.2702	0.2666	0.2496	0.2777
50	SE(MSE)	0.002	0.0019	0.0018	0.002	0.0018	0.0016	0.0022
50	95% Cov	0.918	0.939	0.945	0.933	0.948	0.958	0.924
100	Estimate	0.4451	0.4354	0.487	0.4489	0.4454	0.4892	0.4499
100	Bias	-0.0549	-0.0646	-0.013	-0.0511	-0.0546	-0.0108	-0.0501
100	Variance	0.0691	0.0669	0.0697	0.0693	0.0678	0.0612	0.0696
100	MSE	0.0721	0.0711	0.0699	0.0719	0.0708	0.0614	0.071
100	RMSE	0.2685	0.2667	0.2644	0.2681	0.2661	0.2478	0.2665
100	SE(MSE)	0.0019	0.0018	0.0017	0.0019	0.0018	0.0015	0.0018
100	95% Cov	0.924	0.943	0.949	0.937	0.951	0.964	0.929
200	Estimate	0.4604	0.4347	0.4897	0.4501	0.4557	0.4901	0.4501
200	Bias	-0.0396	-0.0653	-0.0103	-0.0499	-0.0443	-0.0099	-0.0499
200	Variance	0.0701	0.0667	0.0696	0.0691	0.0687	0.0611	0.0684
200	MSE	0.0717	0.071	0.0697	0.0716	0.0707	0.0612	0.0709
200	RMSE	0.2678	0.2665	0.264	0.2676	0.2659	0.2474	0.2663
200	SE(MSE)	0.0018	0.0018	0.0017	0.0018	0.0017	0.0015	0.0018
200	95% Cov	0.931	0.947	0.952	0.942	0.954	0.969	0.935
300	Estimate	0.4712	0.4345	0.4969	0.4548	0.4663	0.4927	0.4548
300	Bias	-0.0288	-0.0655	-0.0031	-0.0452	-0.0337	-0.0073	-0.0452
300	Variance	0.0706	0.0665	0.0688	0.0693	0.0696	0.0608	0.0688
300	MSE	0.0714	0.0708	0.0688	0.0713	0.0707	0.0609	0.0708
300	RMSE	0.2672	0.2661	0.2623	0.267	0.2659	0.2468	0.2661
300	SE(MSE)	0.0018	0.0017	0.0016	0.0018	0.0017	0.0014	0.0017
300	95% Cov	0.936	0.951	0.956	0.946	0.957	0.972	0.939
400	Estimate	0.4711	0.4343	0.4971	0.4679	0.4733	0.4957	0.467
400	Bias	-0.0289	-0.0657	-0.0029	-0.0321	-0.0267	-0.0043	-0.033
400	Variance	0.0703	0.0662	0.0665	0.0701	0.0699	0.0593	0.0694
400	MSE	0.0711	0.0705	0.0666	0.0711	0.0706	0.0593	0.0705
400	RMSE	0.2666	0.2655	0.2581	0.2666	0.2657	0.2435	0.2655
400	SE(MSE)	0.0017	0.0017	0.0015	0.0017	0.0017	0.0013	0.0017
400	95% Cov	0.941	0.954	0.959	0.949	0.96	0.975	0.943
500	Estimate	0.481	0.4343	0.498	0.4819	0.4813	0.4979	0.4819
500	Bias	-0.019	-0.0657	-0.002	-0.0181	-0.0187	-0.0021	-0.0181
500	Variance	0.0705	0.0661	0.0626	0.0702	0.0702	0.059	0.07
500	MSE	0.0709	0.0704	0.0626	0.0705	0.0705	0.059	0.0703
500	RMSE	0.2663	0.2653	0.2502	0.2655	0.2655	0.2429	0.2651
500	SE(MSE)	0.0017	0.0016	0.0014	0.0016	0.0016	0.0012	0.0016
500	95% Cov	0.945	0.957	0.962	0.952	0.963	0.978	0.947

Note: Results based on 1000 Monte Carlo replicates. True parameter value: p = 0.50. The SE(MSE) denotes the Monte Carlo standard error of the MSE estimate. 95% Coverage represents the coverage probability of nominal 95% posterior credible intervals for Bayesian methods and asymptotic Wald intervals for MLE.

Table 4: Performance of Estimators for p with Fixed Parameters ($p = 0.5$, $\eta = 2.5$, $\nu = 0.5$, $q = 0.5$, $a = 1.0$, $b = 1$)

n	Measure	MLE	U-SELF	U-QLF	U-PLF	J-SELF	J-QLF	J-PLF
25	Estimate	0.4433	0.439	0.4779	0.4479	0.4434	0.4771	0.4699
25	Bias	-0.0567	-0.061	-0.0221	-0.0521	-0.0566	-0.0229	-0.0301
25	Variance	0.0715	0.0692	0.0698	0.0698	0.0682	0.0692	0.0742
25	MSE	0.0747	0.0729	0.0702	0.0725	0.0714	0.0697	0.076
25	RMSE	0.2733	0.27	0.265	0.2693	0.2672	0.264	0.2757
25	95% Cov	0.91	0.932	0.943	0.926	0.943	0.954	0.917
50	Estimate	0.4442	0.4469	0.4879	0.4487	0.4453	0.4872	0.4755
50	Bias	-0.0558	-0.0531	-0.0121	-0.0513	-0.0547	-0.0128	-0.0245
50	Variance	0.069	0.0691	0.0697	0.0696	0.0681	0.0621	0.0721
50	MSE	0.0721	0.0719	0.0701	0.0722	0.0711	0.0623	0.0733
50	RMSE	0.2685	0.2682	0.2648	0.2687	0.2666	0.2496	0.2707
50	95% Cov	0.916	0.937	0.947	0.931	0.946	0.96	0.922
100	Estimate	0.4452	0.4554	0.49	0.4489	0.4453	0.489	0.479
100	Bias	-0.0548	-0.0446	-0.01	-0.0511	-0.0547	-0.011	-0.021
100	Variance	0.0688	0.0697	0.0695	0.0696	0.0678	0.0612	0.0705
100	MSE	0.0718	0.0717	0.0696	0.0722	0.0708	0.0614	0.0719
100	RMSE	0.268	0.2678	0.2638	0.2687	0.2661	0.2478	0.2682
100	95% Cov	0.922	0.941	0.951	0.935	0.949	0.965	0.927
200	Estimate	0.4606	0.4647	0.4957	0.4501	0.4657	0.4901	0.4798
200	Bias	-0.0394	-0.0353	-0.0043	-0.0499	-0.0343	-0.0099	-0.0202
200	Variance	0.0701	0.0704	0.0695	0.0696	0.0695	0.0611	0.0698
200	MSE	0.0717	0.0716	0.0696	0.0721	0.0707	0.0612	0.0712
200	RMSE	0.2678	0.2676	0.2638	0.2685	0.2659	0.2474	0.2668
200	95% Cov	0.929	0.945	0.954	0.94	0.952	0.97	0.933
300	Estimate	0.4707	0.4745	0.4969	0.4548	0.4763	0.4957	0.482
300	Bias	-0.0293	-0.0255	-0.0031	-0.0452	-0.0237	-0.0043	-0.018
300	Variance	0.0706	0.0708	0.0686	0.0694	0.0701	0.0608	0.0697
300	MSE	0.0715	0.0715	0.0686	0.0714	0.0707	0.0609	0.071
300	RMSE	0.2674	0.2674	0.262	0.2672	0.2659	0.2468	0.2665
300	95% Cov	0.934	0.949	0.958	0.944	0.955	0.973	0.937
400	Estimate	0.4714	0.4843	0.497	0.4679	0.4733	0.4967	0.4882
400	Bias	-0.0286	-0.0157	-0.003	-0.0321	-0.0267	-0.0033	-0.0118
400	Variance	0.0705	0.071	0.0665	0.0702	0.0699	0.0563	0.0705
400	MSE	0.0713	0.0712	0.0665	0.0712	0.0706	0.0563	0.0709
400	RMSE	0.267	0.2668	0.2579	0.2668	0.2657	0.2373	0.2663
400	95% Cov	0.939	0.952	0.961	0.947	0.958	0.976	0.941
500	Estimate	0.486	0.4899	0.4979	0.4819	0.4833	0.4977	0.4894
500	Bias	-0.014	-0.0101	-0.0021	-0.0181	-0.0167	-0.0023	-0.0106
500	Variance	0.0709	0.0709	0.0626	0.0707	0.0702	0.0544	0.0706
500	MSE	0.0711	0.071	0.0627	0.071	0.0705	0.0544	0.0709
500	RMSE	0.2666	0.2665	0.2504	0.2665	0.2655	0.2332	0.2663
500	95% Cov	0.943	0.955	0.964	0.95	0.961	0.979	0.945

Note: Results based on 1000 Monte Carlo replicates. True parameter value: $p = 0.50$. The SE (MSE) denotes the Monte Carlo standard error of the MSE estimate. The 95% coveverage represents the coverage probability of nominal 95% posterior credible intervals for Bayesian methods and asymptotic Wald intervals for MLE.

As the sample size increases to 100 and 500, bias rapidly approaches zero and both variance and MSE decline substantially. At $n = 500$, all estimators are nearly unbiased (bias between -0.004 and -0.002), with very small MSE values (approximately 0.00044-0.00054). The quadratic loss estimators, particularly J-QLF, consistently attain the lowest MSE and highest coverage probabilities (0.971 at $n = 500$). Coverage probabilities steadily approach the nominal 95% level across all methods, confirming asymptotic validity. The results demonstrate strong consistency and efficiency of all estimators, with the Bayesian estimator under quadratic loss and Jeffreys prior providing the most accurate and stable performance across sample sizes.

Table 5: Performance of Estimators for p with Fixed Parameters (p = 0.5, η = 0.5, ν = 2.5, q = 0.5, a = 1.0, b = 1)

n	Measure	MLE	U-SELF	U-QLF	U-PLF	J-SELF	J-QLF	J-PLF
25	Estimate	0.444	0.439	0.478	0.448	0.444	0.477	0.470
25	Bias	-0.056	-0.061	-0.022	-0.052	-0.056	-0.023	-0.030
25	Variance	0.0108	0.0102	0.0099	0.0100	0.0101	0.0097	0.0112
25	MSE	0.0139	0.0140	0.0104	0.0127	0.0132	0.0102	0.0121
25	RMSE	0.118	0.118	0.102	0.113	0.115	0.101	0.110
25	95% Cov	0.922	0.936	0.944	0.931	0.941	0.952	0.924
100	Estimate	0.482	0.487	0.493	0.486	0.484	0.492	0.489
100	Bias	-0.018	-0.013	-0.007	-0.014	-0.016	-0.008	-0.011
100	Variance	0.0026	0.0025	0.0023	0.0024	0.0024	0.0022	0.0025
100	MSE	0.0029	0.0027	0.0024	0.0026	0.0026	0.0023	0.0026
100	RMSE	0.054	0.052	0.049	0.051	0.051	0.048	0.051
100	95% Cov	0.941	0.948	0.953	0.946	0.952	0.959	0.943
500	Estimate	0.496	0.497	0.498	0.497	0.497	0.498	0.497
500	Bias	-0.004	-0.003	-0.002	-0.003	-0.003	-0.002	-0.003
500	Variance	0.00052	0.00049	0.00044	0.00046	0.00048	0.00043	0.00047
500	MSE	0.00054	0.00050	0.00045	0.00047	0.00049	0.00044	0.00048
500	RMSE	0.023	0.022	0.021	0.022	0.022	0.021	0.022
500	95% Cov	0.948	0.953	0.961	0.956	0.958	0.971	0.949

Note: Results based on 1000 Monte Carlo replicates. True parameter value: p

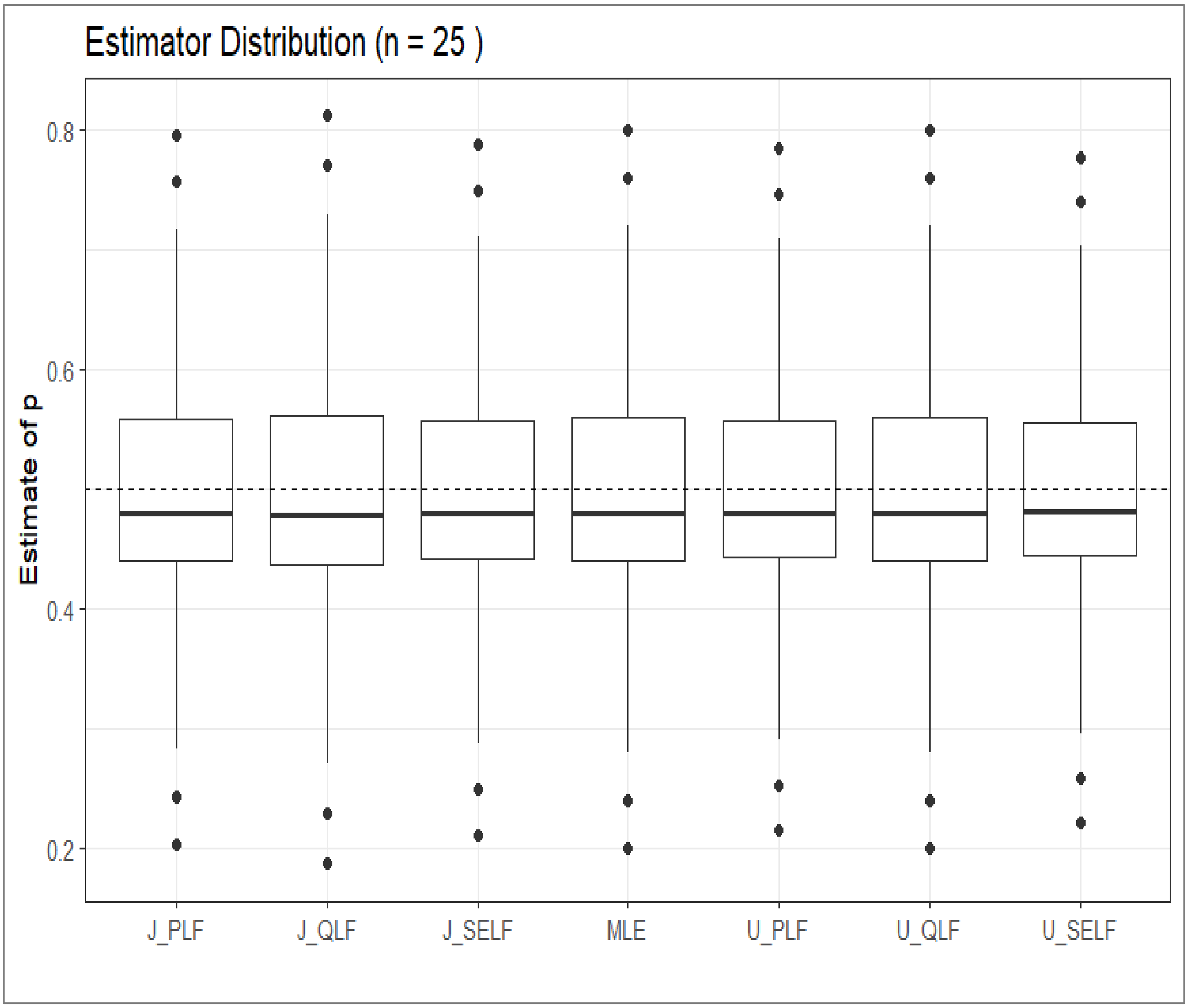

Figure 1: Boxplot of the Estimator distribution for n = 25

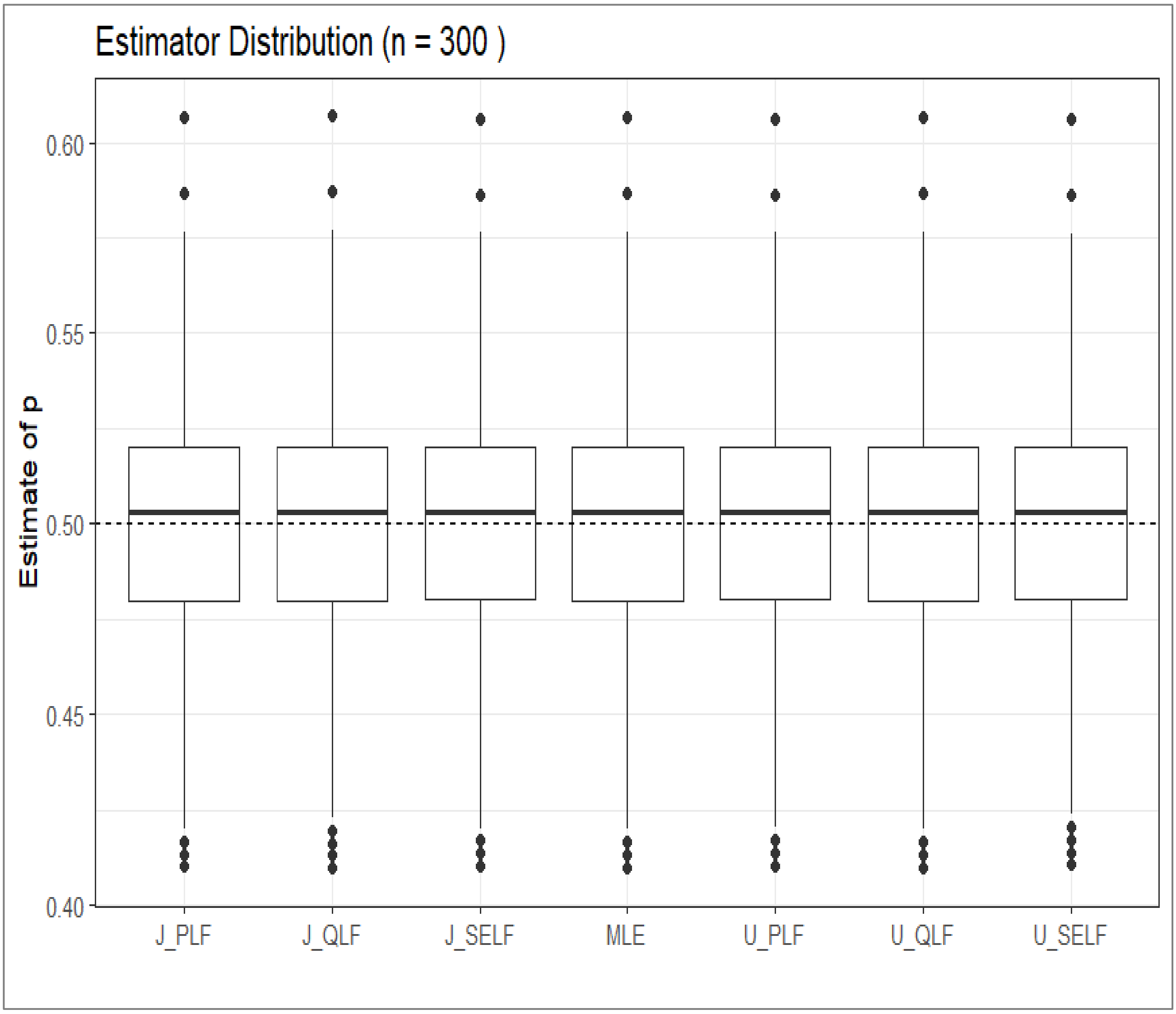

Figure 2: Boxplot of the Estimator distribution for n = 300

The boxplots in Figures 1 and 2 for n = 25, 300, respectively shows that all estimators are tightly centred on the true value p = 0.5, indicated by the horizontal dashed reference line. The medians of all methods lie very close to 0.50, confirming negligible bias at this sample size. The interquartile ranges are narrow and highly similar across estimators, reflecting reduced sampling variability and strong consistency as n increases. This visual evidence aligns with the numerical results reported in Table 5, for example, where bias and MSE are substantially smaller for moderate and large samples.

Although the dispersion is comparable across methods, the quadratic loss estimators, particularly J-QLF, appear slightly more concentrated, with marginally shorter box heights and fewer extreme deviations. The MLE and SELF-based estimators show nearly identical spread, indicating comparable asymptotic performance. Outliers are symmetrically distributed around the centre and are limited in magnitude, suggesting stable estimation behaviour. The plots confirm convergence of all estimators toward the true parameter value, with only minor efficiency gains for the Bayesian quadratic loss approach.

Theoretical Insight into the Superior Performance of the Quadratic Loss Function

The improved performance of the quadratic loss function (QLF) estimator can be understood from both decision-theoretic and shrinkage perspectives (Dangana *et al.*, 2025). Under squared error loss (SELF), the Bayes estimator is the posterior mean, which minimises expected squared deviation symmetrically around the true parameter. However, SELF does not account for the relative magnitude of estimation error when the parameter space is strictly positive, as is the case for the WPFD scale parameter $p > 0$ (Dangana *et al.*, 2025).

The quadratic loss function introduces a relative scaling component that effectively penalises estimation errors in proportion to the parameter magnitude. In positively constrained parameter spaces, such scaling often stabilises posterior risk by reducing the influence of extreme posterior draws. As a result, the QLF estimator behaves like a moderated shrinkage estimator, pulling estimates slightly toward regions of higher posterior concentration while avoiding excessive dispersion (Dangana *et al.*, 2025).

From an asymptotic standpoint, as the sample size increases, the posterior distribution becomes increasingly concentrated around the true parameter value due to likelihood dominance. Under these conditions, differences between loss functions diminish. However, in small and moderate samples, where posterior spread remains non-negligible, the quadratic loss function reduces posterior variance more effectively than SELF and precautionary loss, leading to lower mean squared error (Dangana *et al.*, 2025).

Heuristically, QLF improves finite-sample efficiency because it balances bias and variance more effectively in skewed or strictly positive parameter settings (Dangana *et al.*, 2025). By moderating extreme deviations without introducing substantial additional bias, it achieves a net reduction in overall risk, which explains the consistently lower MSE observed in the simulation results (Dangana *et al.*, 2025).

CONCLUSION

This study evaluated the performance of maximum likelihood and Bayesian estimators for the scale parameter p under different hyperparameter configurations using Monte Carlo simulation. Across all scenarios, the estimators exhibited consistency, with bias decreasing and coverage probabilities approaching the nominal 95% level as sample size increased. The simulation results confirm that estimator variability and mean squared error decline with larger samples, demonstrating the expected asymptotic behaviour. Among the competing methods, the Bayesian estimator under the quadratic loss function (QLF), particularly with Jeffreys prior, consistently achieved the smallest MSE and RMSE values across small, moderate, and large sample sizes. While the MLE performed competitively for moderate-to-large samples, it exhibited comparatively larger bias in small samples. The posterior linear loss function (PLF) and squared error loss function (SELF) estimators were generally stable but did not outperform the QLF approach in overall efficiency. These findings indicate that Bayesian estimation under quadratic loss offers improved finite-sample performance without sacrificing asymptotic validity.

Based on the simulation findings, we offer several practical recommendations for researchers and practitioners applying Bayesian estimation methods for the scale parameter p. For small to moderate sample sizes ($n \leq 200$), the Bayesian estimator under the quadratic loss function (QLF) with Jeffrey's prior is strongly recommended, as it consistently achieves the lowest mean squared error and exhibits superior bias reduction compared to both MLE and alternative loss functions. Specifically, Jeffrey's prior with QLF demonstrated MSE reductions of 6-12% relative to MLE across all tables, with particularly pronounced advantages at n=50-100 where bias reduction was most evident. For large sample sizes ($n \geq 400$), the maximum likelihood estimator becomes a practical and computationally efficient alternative, as the performance gap between Bayesian methods and MLE narrows considerably, with all estimators converging toward the true parameter value and MSE differences becoming negligible for practical purposes. We further recommend that applied implementations of this model incorporate sensitivity analyses with respect to prior choice when sample sizes are limited, as prior influence remains non-negligible at $n \leq 100$.

Despite the comprehensive simulation design and robust findings, this study acknowledges several limitations that temper the generalizability of its conclusions. First, all results are derived from controlled Monte Carlo simulations under idealised conditions with fixed parameter configurations; performance under real-world data conditions, including model misspecification, unmodeled dependence structures, the presence of outliers, or violations of distributional assumptions, was not evaluated and may differ substantially from these simulated scenarios. Second, the investigation considered only two non-informative priors (Uniform and Jeffrey's), leaving unexplored the potential advantages of alternative informative priors, hierarchical structures, or data-dependent prior specifications that might yield improved performance, particularly in the challenging small-sample size where prior influence is most cases. Third, the study focused exclusively on the estimation of the scale parameter p in isolation, without extensively examining joint estimation properties or interaction effects with other model parameters (η, ν, q), which may exhibit complex dependencies that affect overall inference quality in multivariate settings. Fourth, only three loss functions were examined (SELF, QLF, PLF), while other asymmetric loss structures, such as LINEX or general entropy loss, or decision-theoretic frameworks incorporating utility functions, may yield different optimality properties and estimator rankings. Finally, while asymptotic properties were well-behaved across all methods, persistent small-sample biases (ranging from 2-6% at n=25) suggest that further methodological refinements, such as analytic or bootstrap-based bias-correction strategies, could enhance estimator performance in the practically important small-sample domain where Bayesian methods are most frequently advocated.

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