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ORIGINAL RESEARCH ARTICLE

The Generalized Gompertz-G Family of Distributions: Statistical Properties and Applications

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ABSTRACT
This research aimed at presenting a new statistical model called the Generalized Gompertz-G family of distribution via the method of Alzaatreh, which introduces additional shape parameters for any baseline distribution. We investigate various mathematical aspects of this model, explicitly deriving properties such as moments, moment-generating function, survival function, hazard function, entropies, quantile function, and order statistics distribution. We explore a particular member of this family of distributions, the Generalized Gompertz-Exponential Distribution (GGED), by defining its properties and doing a detailed analysis. A Monte Carlo simulation was utilized to evaluate the model's flexibility and performance, and the distribution family's potential utility in real-world data analysis was further highlighted by investigating the model's parameter estimation using the method of maximum likelihood. We also assess the adaptability of the Generalized Gompertz-Exponential distribution using a real-life dataset and relating its performance with other established models through information criterion. The results show that the Generalized Gompertz-Exponential distribution (GGED) outperformed the compared distributions, emphasizing its potential applicability in diverse practical scenarios for data modeling.

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Gompertz distribution, Generalized Gompertz-G Family, Exponential distribution, Quantile function, Maximum likelihood.

INTRODUCTION

The Gompertz distribution has garnered significant attention and widespread use by researchers in various fields over the past few decades. This distribution was formulated to model human mortality, survival times, and actuarial tables (Gompertz, 1825). Gompertz distribution has found applications in a wide range of disciplines, such as biological studies, reliability analysis, medical sciences, actuarial sciences, engineering, environmental science, demography, economics, and finance (Alizadeh *et al.,* 2017). Essentially, it is a broadened form of the exponential distribution and is frequently utilized in examining lifespan studies (Sanku *et al.,* 2018). Researchers have devised numerous additional distributions to better capture complex datasets by developing a new family of probability distributions that attracted the attention of dedicated scholars and statisticians who value the flexibility offered by these distributions. Notable examples encompass the Exponentiated-G (E-G) class introduced by Gupta *et al.* (1998), the Beta-G class by Eugene *et al.* (2002), the Gamma-G distributions by Zografos and Balakrishnan (2009), and the Kumaraswamy Weibull-G family by Cordeiro *et al.,* (2010), the Generalized Gompertz distribution by El-Gohary *et al.,* (2013), the Weibull-G family of Bourguignon *et al.,* (2014), the Kumaraswamy-G family by Cordeiro and Castro (2011), the Power Gompertz distribution by Ieren *et al.,* (2019), Type I Half-Logistic Exponentiated-G Family by Bello *et al.,* (2021), the Transform-Transformer family introduced by Alzaatreh *et al.,* (2013), New Generalized Weibull Odd Frechet Family by Usman *et al.,*(2020), Kumaraswamy-Odd Rayleigh-G by Falgore and Doguwa (2020), Odd Gompertz- G Family by Kajuru *et al.,* (2023) and New Generalized Odd Frechet-G by Abubakar Sadiq *et al.,*(2023). Yet, some real datasets cannot fit so many existing distributions. This provides an avenue for developing a new family of distributions that can be used to generalize any parent distribution that will accommodate diverse behavior patterns in practical applications to modeling different datasets. The tractability of a probability distribution is crucial since more adaptable models provide richer evidence compared to less flexible ones. However, the manageability of these distributions also matters, especially when generating

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random samples (Usman *et al.,* 2020). However, Alizadeh *et al.* (2017) developed the Gompertz-G (G-G) family of distribution using a link function without an extra shape parameter in the upper limit of the CDF of the random variable X. Therefore, this research aimed to develop a new generator using a link function with an extra shape parameter in the upper limit of the CDF of the random variable X known as the Generalized Gompertz-G (GG-G) family of distributions to accommodate more flexible distributions in fitting real-life datasets and displaying different behavioral shapes in practical applications compared to other class of distributions.

MATERIAL AND METHOD

Generalized Gompertz-G (GG-G) Family

The cumulative distribution function (CDF) and probability density function (PDF) of the Gompertz distribution, as described by Lanert (2012), with θ representing the scale parameter and γ as the shape parameter, can be expressed in the following comportment:

$$G(t;\theta,\gamma) = 1 - e^{-\frac{\theta}{\gamma}(e^{\gamma t}-1)}; 0 < t < \infty, \theta, \gamma > 0 \quad (1)$$

$$g(t;\theta,\gamma) = \theta e^{\gamma t} e^{-\frac{\theta}{\gamma}(e^{\gamma t}-1)}; 0 < t < \infty, \theta, \gamma > 0 \quad (2)$$

Let $g(t;\theta,\gamma)$ be the PDF of the Gompertz distribution and let $F(x)$ be the CDF of a random variable X. We defined the CDF of the Generalized Gompertz-G (GG-G) family of distribution by integrating the density function of equation (2) to attain the CDF as:

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = \int_0^{-\log[1-G(x;\Phi)^{\alpha}]} g(t;\theta,\gamma)dt = 1 - e^{-\frac{\theta}{\gamma}[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} \quad (3)$$

The corresponding PDF is realized by differentiating equation (3) as follows:

$$f_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = \alpha\theta g(x;\Phi)G(x;\Phi)^{\alpha-1}[1 - G(x;\Phi)^{\alpha}]^{-\gamma-1} e^{-\frac{\theta}{\gamma}[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} \quad (4)$$

Where $\theta > 0$ is the scale parameter, and $\alpha > 0, \gamma > 0$ are the shape parameters, $g(x,\Phi)$ and $G(x;\Phi)$ are the PDF and CDF of any parent distribution, and Φ is the parameter vector in the model, therefore a random variable X with distribution function and density function in equations (3) and (4) is denoted by $X \sim GG - G(\alpha,\theta,\gamma,\Phi)$.

Survival and Hazard Rate Function of the GG-G Family

The survival function and hazard function are provided as follows, respectively:

$$S_{GG-G}(x) = e^{-\frac{\theta}{\gamma}[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} \quad (5)$$

$$h_{GG-G}(x) = \alpha\theta g G(x;\Phi)^{\alpha-1}[1 - G(x;\Phi)^{\alpha}]^{-\gamma-1} \quad (6)$$

Quantile Function of GG-G Family

The quantile function of the GG-G family is derived by reversing the cumulative distribution function (CDF) provided in equation (3), it is stated as:

$$x = Q(u) = G^{-1} \frac{\frac{1}{\gamma}\log\left(1-\frac{\gamma}{\theta}\log(1-u)\right)}{\left[1+\left[\frac{1}{\gamma}\log\left(1-\frac{\gamma}{\theta}\log(1-u)\right)\right]\right]} \quad (7)$$

In this context, G^{-1} represents the quantile function of a continuous parent distribution, and "u" is treated as a random variable following a uniform distribution in the range of (0, 1).

Reduced form of the CDF and PDF of GG-G Family

In this section, we will delve into an insightful expansion of the cumulative distribution function of the GG-G family.

Lemma: The equation that offers a linear representation of the GG-G family of distributions can be stated as follows:

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1 - \sum_{k=0}^{\infty} Y_k B_{\alpha k}(x;\Phi) \quad (8)$$

Proof

$$F(x;\alpha,\theta,\gamma,\Phi) = 1 - e^{-\frac{\theta}{\gamma}[(1-G(x)^{\alpha})^{-\gamma}-1]}$$

From the expansion using a power series

$$e^{-\frac{\theta}{\gamma}[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\left(\frac{\theta}{\gamma}\right)^i [(1 - G(x;\Phi)^{\alpha})^{-\gamma}]^i$$

And $$[(1 - G(x;\Phi)^{\alpha})^{-\gamma}]^i = \left[\frac{1}{(1-G(x;\Phi)^{\alpha})^{\gamma}} - 1\right]^i = \left[\frac{1-(1-G(x;\Phi)^{\alpha})^{\gamma}}{(1-G(x;\Phi)^{\alpha})^{\gamma}}\right]^i$$

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1 - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\left(\frac{\theta}{\gamma}\right)^i [1 - (1 - G(x;\Phi)^{\alpha})^{\gamma}]^i [(1 - G(x;\Phi)^{\alpha})^{-\gamma}]^i \quad (9)$$

By binomial expansion,

$$[1 - (1 - G(x;\Phi)^{\alpha})^{\gamma}]^i = \sum_{j=0}^{\infty}(-1)^j \binom{i}{j} (1 - G(x;\Phi)^{\alpha})^{\gamma j}$$

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{i!}\left(\frac{\theta}{\gamma}\right)^i \binom{i}{j} (1 - G(x;\Phi)^{\alpha})^{\gamma j}(1 - G(x;\Phi)^{\alpha})^{-\gamma i}$$

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1 - \theta^i\gamma^{-i} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{i}{j} (1 - G(x;\Phi)^{\alpha})^{-\gamma(i-j)} \quad (10)$$

By binomial expansion,

$$(1-G(x)^{\alpha})^{-\gamma(i-j)} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma(i-j))}{k!\,\Gamma\gamma(i-j)} (G(x;\Phi)^{\alpha})^{k}$$

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\theta^{i}\gamma^{-i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}\Gamma(k+\gamma(i-j))}{i!k!\Gamma\gamma(i-j)}\binom{i}{j}G(x;\Phi)^{\alpha k} \quad (11)$$

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{k=0}^{\infty} Y_k\, G(x;\Phi)^{\alpha k}$$

$$F_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{k=0}^{\infty} Y_k\, B_{\alpha k}(x;\Phi) \quad (12)$$

Where $Y_k = \theta^{i}\gamma^{-i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(k+\gamma(i-j))}{i!k!\Gamma\gamma(i-j)}\binom{i}{j}$ and $B_{(\alpha k)}(x;\Phi) = G(x;\Phi)^{\alpha k}$ denotes the CDF of the Exponentiated-G distribution with power parameter $\alpha k > 0$. We can express the probability density function of X in a linear mixture of Expt-G density function as follows:

$$f_{GG-G}(x;\alpha,\theta,\gamma,\Phi) = \sum_{k=0}^{\infty} Y_k b_{\alpha k}(x;\Phi) \quad (13)$$

where $b_{\alpha k}(x;\Phi) = \alpha k g(x;\Phi) G(x;\Phi)^{\alpha k-1}$

Moments of the GG-G Family

The expression for the r^th moment of a random variable X, which adheres to the GG-G family is as follows:

$$E(X^r) = \int_0^{\infty} x^r f_{GG-G}(x;\alpha,\theta,\gamma,\Phi)dx = \int_0^{\infty} x^r \sum_{k=0}^{\infty} Y_k b_{\alpha k}(x;\Phi)dx = \sum_{k=0}^{\infty} Y_k E(X^r{}_k) \quad (14)$$

Where $E(X^r{}_k) = \int_0^{\infty} x^r b_{\alpha k}(x;\Phi)dx$

Moment-Generating Function of the GG-G Family

The moment-generating function for a random variable X, which belongs to the GG-G family, is expressed as:

$$M_x{}^{GG-G}(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f_{GG-G}(x;\alpha,\theta,\gamma,\Phi)dx = \sum_{k=0}^{\infty} Y_k E(e^{tx_k}) \quad (15)$$

Where $E(e^{tx_k}) = \int_0^{\infty} e^{tx} b_{\alpha k}(x;\Phi)dx$

Entropy of the GG-G Family

Entropy serves as a metric to gauge the level of diversity or unpredictability in a random variable X. Statistically, the (Renyi, 1961) entropy of the GG-G family is defined as follows:

$$I_R(\Psi) = \frac{1}{1-\Psi}\log\int_0^{\infty} f_{GG-G}(x;\alpha,\theta,\gamma,\Phi)^{\Psi}dx$$

$$= \frac{1}{1-\Psi}\log\int_0^{\infty}\left(\sum_{k=0}^{\infty} Y_k b_{\alpha k}(x;\Phi)\right)^{\Psi}dx$$

$$I_R(\Psi) = \frac{1}{1-\Psi}\log\left(\sum_{k=0}^{\infty} Y_k\right)^{\Psi}\int_0^{\infty}(b_{\alpha k}(x;\Phi))^{\Psi}dx \quad (16)$$

Where $\Psi > 0$ and $\Psi \neq 1$

Order Statistics of the GG-G Family

Suppose we have a random sample with values $X_1, X_2, \ldots, X_n$ from the GG-G distribution, and we denote the corresponding order statistics as $X_{1:n} \leq X_{2:n} \leq \ldots X_{n:n}$. In this context, the expression for the i^{th} order statistic can be stated as:

$$f_{i:n}(x;\alpha,\theta,\gamma,\Phi) = \frac{n!}{(i-1)(n-i)!}[f(x;\alpha,\theta,\gamma,\Phi)][F(x;\alpha,\theta,\gamma,\Phi)]^{i-1}[1-F(x;\alpha,\theta,\gamma,\Phi)]^{n-i}$$

$$= \frac{n!}{(i-1)(n-i)!}\left[\sum_{k=0}^{\infty} Y_k h_{\alpha k}(x;\Phi)\right]\left[1-\sum_{k=0}^{\infty} Y_k B_{\alpha k}(x;\Phi)\right]^{i-1}\left[\sum_{k=0}^{\infty} Y_k B_{\alpha k}(x;\Phi)\right]^{n-i} \quad (17)$$

Estimation of Parameters of the GG-G Family

Assuming we have $x_1, x_2, x_3, \ldots, x_n$ observed values from the proposed GG-G family with α, θ, γ parameters, and we have a $[m \times 1]$ parameter vector. The log-likelihood function, denoted as Ψ is formulated as follows:

$$L(\psi) = \log\prod_{i=1}^{n} f(x) = n\log\alpha + n\log\theta + \sum_{i=1}^{n}\log(g(x;\Phi)) + (\alpha-1)\sum_{i=1}^{n}\log(G(x;\Phi)) - (\gamma+1)\sum_{i=1}^{n}\log(1-G(x;\Phi)^{\alpha}) - \frac{\theta}{\gamma}\sum_{i=1}^{n}\log[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1] \quad (18)$$

The partial derivatives of equation (18) with respect to the $(\alpha, \theta, \gamma, \Phi)$ parameters are provided as follows:

$$\frac{\partial L(\psi)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log G(x;\Phi) - (\gamma + 1)\sum_{i=1}^{n} \frac{G(x;\Phi)^{\alpha} \ln G(x;\Phi)}{(1-G(x;\Phi)^{\alpha})} + \frac{\theta}{\gamma}\sum_{i=1}^{n} \frac{G(x;\Phi)^{\alpha} \ln G(x;\Phi)}{[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} \quad (19)$$

$$\frac{\partial L(\psi)}{\partial \theta} = \frac{n}{\theta} - \frac{1}{\gamma}\sum_{i=1}^{n} \log[(1-G(x;\Phi)^{\alpha})^{-\gamma} - 1] \quad (20)$$

$$\frac{\partial L(\psi)}{\partial \gamma} = -\sum_{i=1}^{n} [\log(1-G(x;\Phi)^{\alpha})] + \frac{\theta}{\gamma^2}\sum_{i=1}^{n} \log[(1-G(x;\Phi)^{\alpha})^{-\gamma} - 1] + \theta\sum_{i=1}^{n} \frac{(1-G(x;\Phi)^{\alpha})^{-\gamma-1}}{[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} \quad (21)$$

$$\frac{\partial L(\psi)}{\partial \Phi} = \sum_{i=1}^{n} \frac{g'(x;\Phi)}{g(x;\Phi)} + (\alpha-1)\sum_{i=1}^{n} \frac{g(x;\Phi)}{G(x;\Phi)} + (\gamma + 1)\sum_{i=1}^{n} \frac{\alpha g(x;\Phi)G(x;\Phi)^{\alpha-1}}{(1-G(x;\Phi)^{\alpha})} + \frac{\theta}{\gamma}\sum_{i=1}^{n} \frac{\alpha g(x;\Phi)G(x;\Phi)^{\alpha-1}}{[(1-G(x;\Phi)^{\alpha})^{-\gamma}-1]} \quad (22)$$

Sub-Model of the GG-G Family

When you introduce an Exponential distribution into the GG-G family, you generate a new distribution. The cumulative distribution function (CDF) and probability density function (PDF) of the Exponential distribution, which serves as the foundational distribution with a parameterμ, are expressed as follows:

$$P(x;\mu) = 1 - e^{-\mu x} \quad (23)$$

$$p(x;\mu) = \mu e^{-\mu x} x > 0, \mu > 0 \quad (24)$$

Inducing equations (23) and (24) into equations (3), (4), (5), (6), and (7), will provide the distribution function$F(x)$, density function$f(x)$, survival function$S(x)$, hazard function$h(x)$, and quantile function$Q(u)$ of the Generalized Gompertz-Exponential distribution (GG-ED).

Graph for the Sub-Model of the GG-G Family

Here, we display graphs (Figure 1, 2, 3 and 4) depicting the density function, distribution function, survival function, and hazard function of the GG- Exponential distribution across various parameter values;

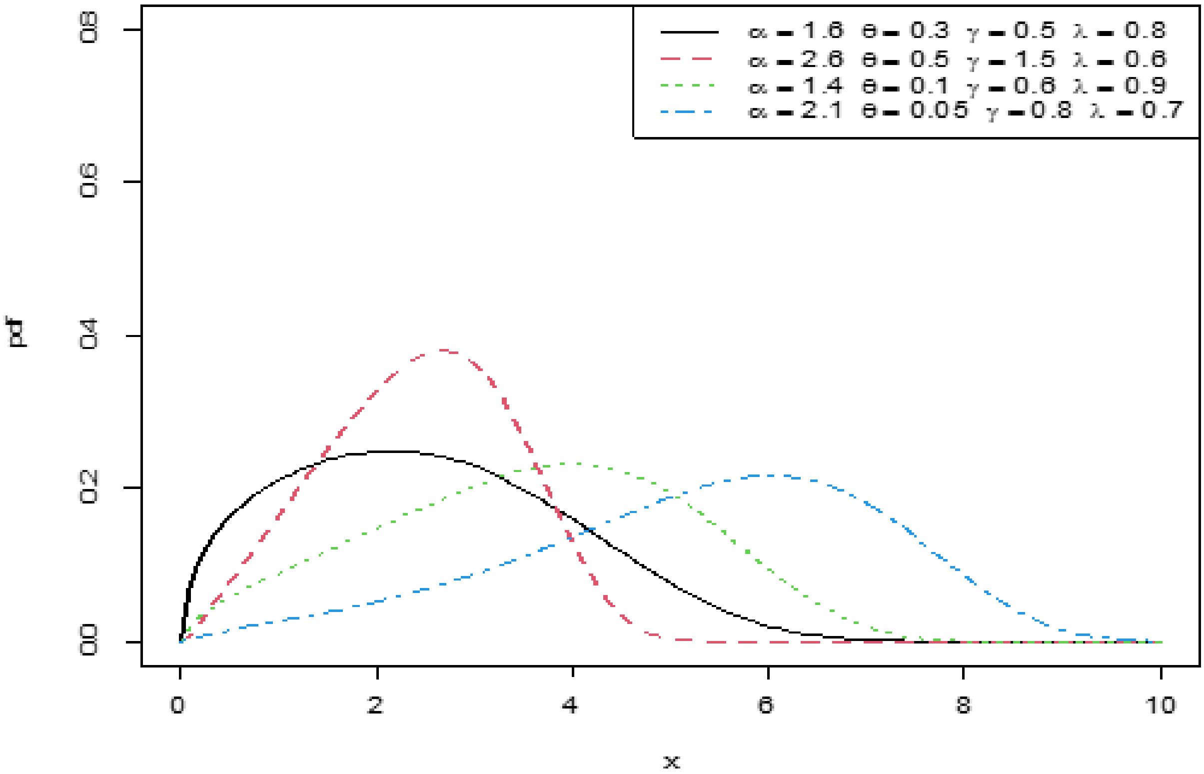

Figure 1: PDF of the Generalized Gompertz-Exponential Distribution

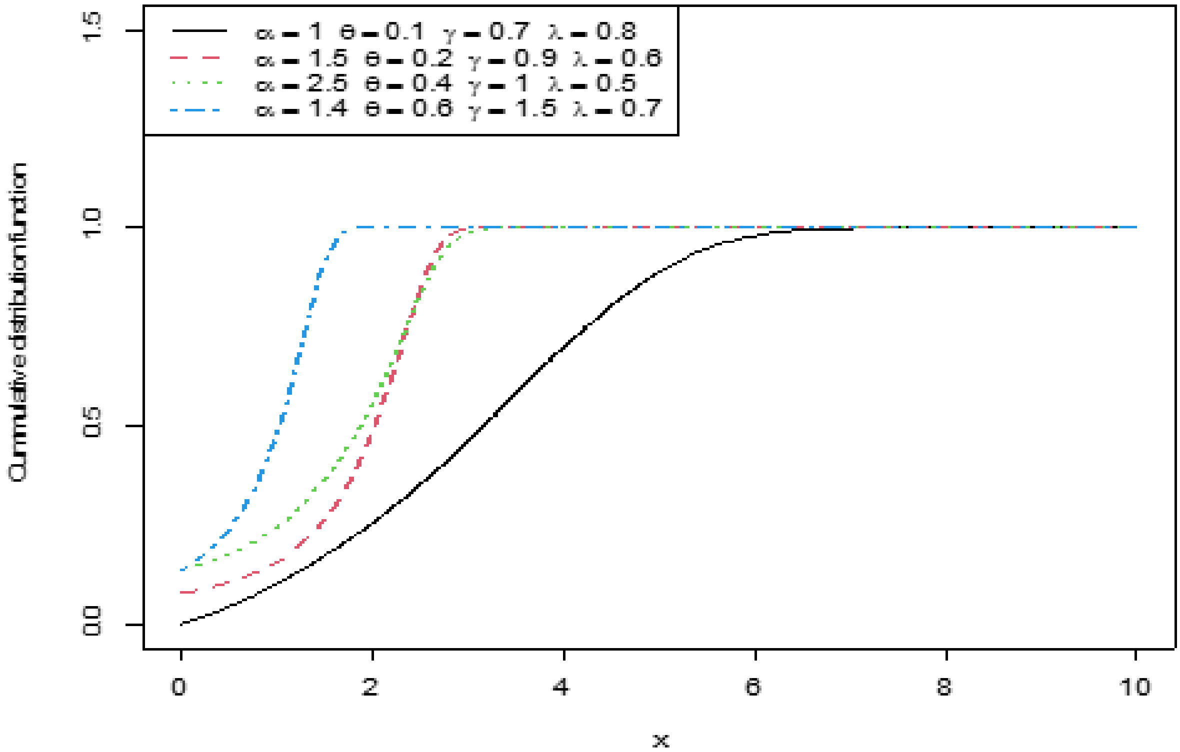

Figure 2: CDF of the Generalized Gompertz-Exponential Distribution

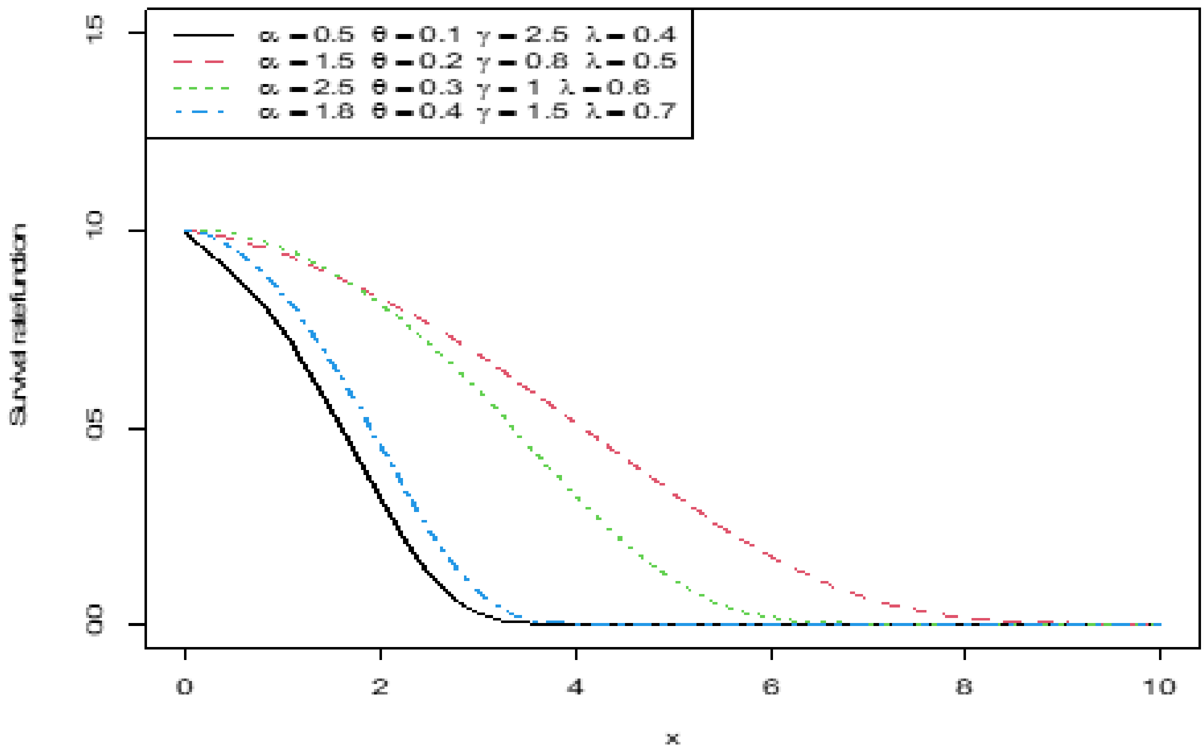

Figure 3: Survival function of the Generalized Gompertz-Exponential Distribution

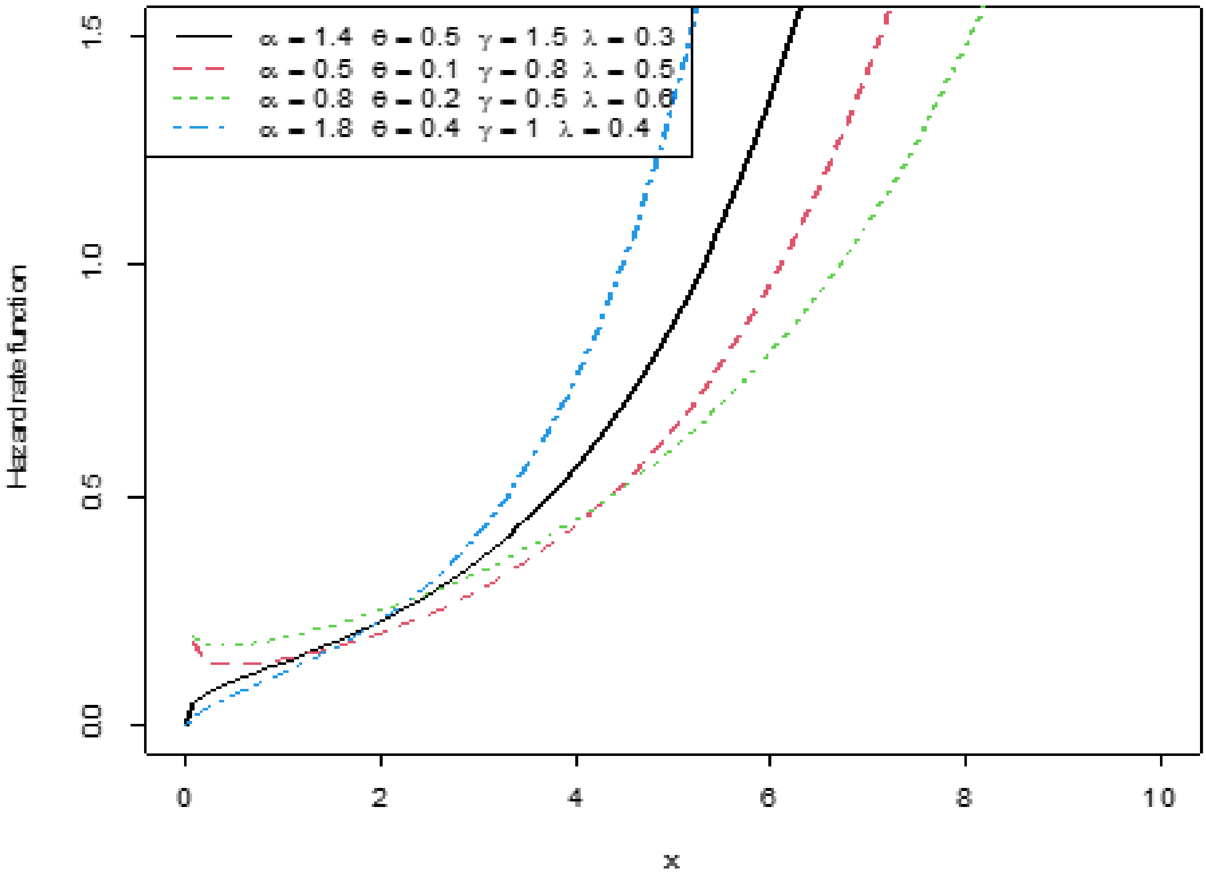

Figure 4: Hazard function of the Generalized Gompertz-Exponential Distribution

MONTE CARLO SIMULATION AND APPLICATION

Monte Carlo Simulation

The widely used category of computational techniques called "Monte Carlo simulation" is utilized to generate numerical outcomes from replicated random samples. This approach is employed to tackle the issue of assessing risk in modeling lifetime data.

Simulation Study

To assess the reliability of the GG-ED, a simulation study was conducted using the Monte Carlo Simulation technique. The objective of this study was to compute the average, bias, and root mean square error of the model parameters estimated through maximum likelihood estimation. Simulated data was created by applying the quantile function defined in equation (7), and this procedure was reiterated 1,000 times across different sample sizes: n = 20, 50, 100, 250, 500, and 1,000. The parameters remained fixed at a specific value for each of these simulation runs.

The result from Table 1 proves that the bias and root mean square errors (RMSEs) become smaller as the sample size increases. This pattern indicates that the estimates are getting closer to the true values, suggesting they are becoming more accurate and dependable. It demonstrates that the estimates exhibit both efficiency and consistency as the sample size increases.

Application

In this context, we demonstrate the applicability of the Generalized Gompertz-Exponential Distribution (GG-ED) using a real dataset obtained from previous research, as outlined in Arshad *et al.* (2021). We computed the maximum likelihood estimates and assessed goodness-of-fit measures using R software. We then compared the results with those of other distributions, namely the Weibull Exponential (WE), Gompertz Exponential (GE), Kumaraswamy Exponential (KE), Exponentiated Weibull-Exponential (EW-E), and Exponential (E) distributions.

In determining the best out of the competing models, the Akaike Information Criterion, Akaike (1974) "AIC" was employed and is statistically expressed as:

$AIC = -2LL + 2K.$ Where "LL" stands for log-likelihood function and K is the number of model parameters in the model.

The data describes the 85 hailing times of civil engineering dataset obtained from Arshad *et al.* (2021) as follows;

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80

Table 1: Results of the simulated data from the GG-ED for some values of parameters

(θ =0.5, α =1.5, λ =0.5, γ =2)				
n	Parameters	Estimates	Bais	RMSE
20	θ	0.5879	0.0879	0.4008
	α	1.7168	0.2168	0.5765
	λ	0.5432	0.0432	0.1301
	γ	2.0389	0.0389	0.5071
50	θ	0.5829	0.0829	0.2878
	α	1.6265	0.1265	0.4159
	λ	0.5083	0.0083	0.0848
	γ	2.0482	0.0482	0.3810
100	θ	0.5712	0.0712	0.2156
	α	1.5898	0.0898	0.2881
	λ	0.4973	-0.0027	0.0558
	γ	2.0522	0.0522	0.2657
250	θ	0.5543	0.0543	0.1650
	α	1.5535	0.0535	0.1899
	λ	0.4929	-0.0071	0.0327
	γ	2.0457	0.0457	0.1721
500	θ	0.5444	0.0444	0.1277
	α	1.5351	0.0351	0.1306
	λ	0.4918	-0.0082	0.0259
	γ	2.0392	0.0392	0.1182
1000	θ	0.5342	0.0342	0.0941
	α	1.5254	0.0254	0.0967
	λ	0.4928	-0.0072	0.0195
	γ	2.0302	0.0302	0.0947

Table 2: Parameters Estimates and Goodness of Fit Measures for civil engineering data with 85 hailing times.

Model	Parameter Estimates and Goodness of Fit					
	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$	-LL	AIC
GGE	8.1706	3.2261	0.2771	3.1506	136.5066	**281.0133**
WE	0.0073	1.0193	0.7737	-	141.584	289.1680
OGE	0.1168	0.1358	2.2030	-	147.5815	301.1631
GE	0.0163	0.8348	0.7680	-	145.4489	296.8978
Ex	0.1757	-	-	-	232.7956	467.5913
EWE	0.8696	0.0068	1.0046	0.7831	143.6837	295.3674

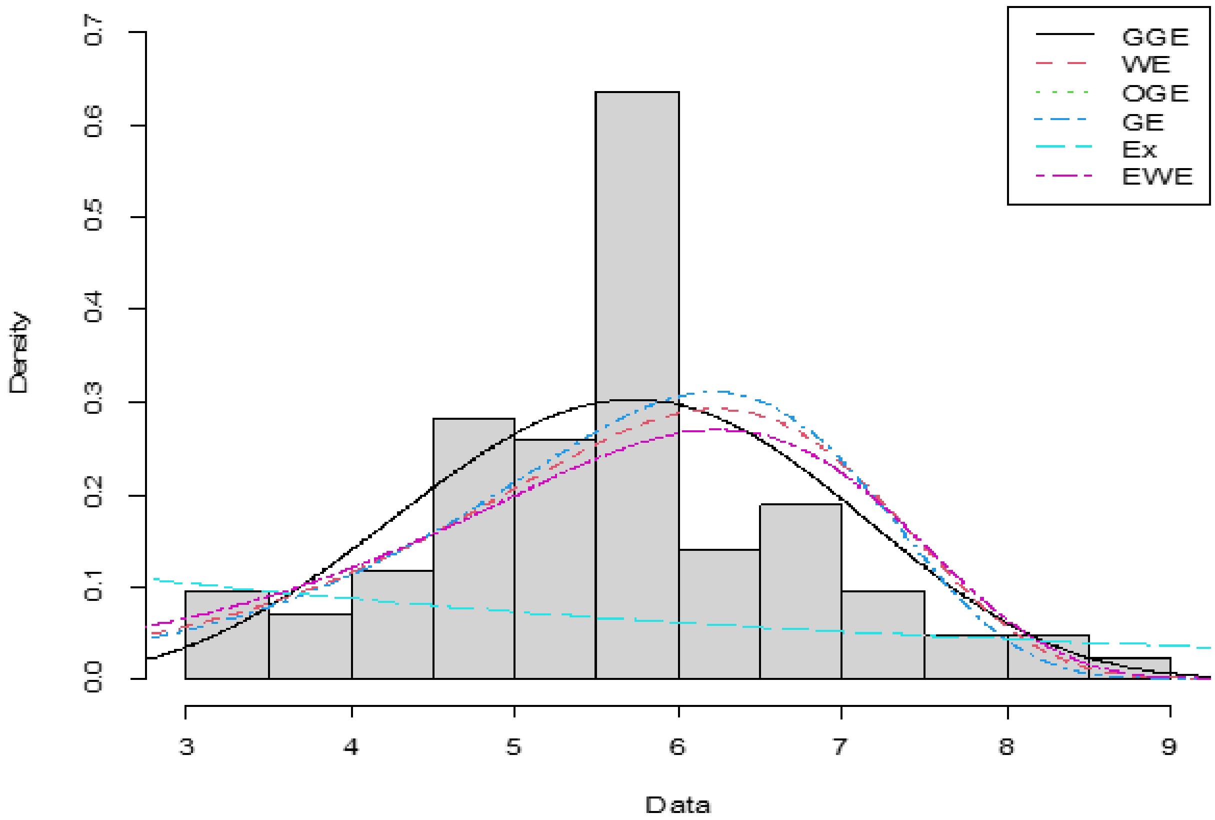

Figure 5: Histogram Plot of the Distributions on the civil engineering data with 85 hailing times

Table 2 showcases the outcomes of maximum likelihood estimation for the parameters of the new distribution and five other reference distributions. When assessing the goodness of fit, it was observed that the proposed distribution GG-ED exhibited the lowest AIC value, with EWE closely following. A visual examination of the fit, depicted in Figure 5, further corroborates that the proposed distribution outperformed the compared distributions. Consequently, among the distributions under consideration, the Generalized Gompertz-Exponential Distribution (GG-ED) is recognized as the most suitable for modeling data related to civil engineering, particularly the 85-hailing times.

CONCLUSION

This paper introduces and investigates a newly developed continuous probability distribution called the Generalized Gompertz-G (GG-G) Family of Distribution. We examined various statistical aspects of this innovative distribution, including the explicit moment, quantile function, entropies, reliability function, hazard function, and order statistics. The distribution parameters were estimated using the maximum likelihood technique. Simulation results were presented to evaluate the performance of this novel distribution. Additionally, we extend the exponential distribution by inducing it to the GG-G, develop the Generalized Gompertz- Exponential distribution (GG-ED), and apply the distribution to analyze a real dataset of 85 hailing times of an engine. The results show that the Generalized Gompertz-Exponential distribution (GG-ED) outperformed the compared distributions, highlighting the distribution's potential utility and applicability in a diverse range of practical scenarios for modeling data related to civil engineering.

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