UMYU Scientifica, Vol. 2 NO. 4, December 2023, Pp 056 – 064
https://doi.org/10.56919/usci.2324.007
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

New Odd Generalized Exponentiated Exponential-G Family of Distributions.

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ARTICLE HISTORY

Received August 14, 2023.
Accepted November 07, 2023.
Published December 21, 2023.

KEYWORDS

Odd generalized exponential-G, T-X family of distribution, quantile function, and Maximum likelihood estimation

ABSTRACT

In the last decades, many researchers have developed new methods for generating families of distributions. These generators are obtained by adding one or more extra shape parameter(s) to the baseline distribution to achieve more flexibility for modelling real lifetime data sets. The additional parameter(s) has been proven useful by obtaining tail properties and improving the analysis from the goodness-of-fit for the families of distributions under study. In this paper, we proposed a new family of distributions called the New Odd Generalized Exponentiated Exponential-G Family of Distributions. The statistical properties are derived, and the maximum likelihood estimation technique is described for the proposed new family of distributions. The new models' performance is illustrated by numerical analysis using a real-life dataset, and the dataset shows that the new models offer a better fit compared to other competing models.

INTRODUCTION

In the last decades, many researchers have delved into various methods for generating new families of distributions termed generators or generalized classes of probability distributions. It is well-known that when a distribution is generalized, an additional shape parameter(s) is added to the baseline distribution. The importance of these extra shape parameter(s) is to vary the tail weight of the resulting compound distribution to be used effectively to describe or analyze skewed data, which provides more flexibility for modelling real lifetime data sets. The extensions sometimes offer better parametric fits to some application aspects and reliability analysis. Some of the recent literatures on generalized families of distributions are: The Kumaraswamy-G family of distributions by Cordeiro and de Castro (2011), Log-gamma-G family of distributions by Amini *et al.* (2012), Exponentiated –G by family of distributions Cordeiro *et al.* (2013), Weibull-G family of distributions by Bourguignon *et al.* (2014), The odd generalized Exponential family of distributions by Tahir *et al.* (2015), Topp-Leone family of distributions by Al-shomrani *et al.* (2016), Topp-Leone G-family of distributions by Rezaei *et al.* (2017), type I generalized exponential class of distributions by Hamedani *et al.* (2018), The Exponentiated Kumaraswamy-G family of distributions by Silva *et al.* (2019), The Fréchet Topp Leone G Family of distributions by Reyad *et al.* (2019a), Transmuted odd Fréchet G by Badr *et al.* (2020), Topp Leone exponentiated-G Family of Distributions by Ibrahim *et al.* (2020), Rayleigh-exponentiated odd generalized-pareto distribution by Yahaya and Doguwa (2021), Topp-Leon Weibull generated family of distributions by Ibrahim *et al.* (2022), A new compound-G Family of distributions Masoom *et al.* (2023).

In this paper, a new two parameter family of distributions is introduced by using the idea of T-X methodology. Some statistical properties of the (NOGEE-G) family were derived and studied.

This paper is organized as follows: in Section 2, we define the cumulative distribution, probability density, reliability, and hazard functions of the new odd generalized exponentiated-exponential-G (NOGEE−G) family of distributions, respectively. Furthermore, we introduce the statistical properties, including the quantile and median are provided in Section 3. Section 4 presents the sub-models of the NOGEE−G family of distributions. Maximum likelihood estimation of the parameters is determined in Section 5. A numerical analysis of NOGEE−G

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How to cite: Odeyale, A. B., Gulumbe, S. U., Umar, U., & Aremu, K. O. (2023). New Odd Generalized Exponentiated Exponential-G Family of Distributions. *UMYU Scientifica*, 2(4), 56 – 64. https://doi.org/10.56919/usci.2324.007

distributions is performed using a real-life data set in Section 6. Conclusion in Section 7.

2. New Odd Generalized Exponentiated Exponential-G Family of Distributions

2.1 The Proposed (NOGEE-G) Family of Distributions

In this work, a new generalized family of distribution is proposed called the New Odd Generalized Exponentiated Exponential-G (NOGEE-G) family of distributions from the cumulative distribution function (cdf) of the Exponentiated-G family of distributions based on Lehmann (1953) defined by as

$$F(x;\theta,\xi)=\left[1-H(x;\xi)\right]^{\theta}, \quad 0\le x<\infty,\ \theta,\xi>0 \tag{1}$$

With the pdf of;

$$f(x;\theta,\xi)=\theta h(x;\xi)\left[1-H(x;\xi)\right]^{\theta-1} \tag{2}$$

Where θ is an exponentiated parameter belonging to a set of positive real numbers.

According to Alzaatreh *et al.* (2013), the cdf of the T-X family of distribution is given as

$$F(x)=\int_{a}^{W[G(x)]} r(t)dt=R\left[W\left[G(x)\right]\right] \tag{3}$$

Where $W\left[G(x)\right]$ satisfies the following conditions

(i) $W\left[G(x)\right]\in\left[a,b\right]$

(ii) $W\left[G(x)\right]$ is differentiable and monotonically non-decreasing, and (4)

(iii) $W[G(x)]\to a$ as $x\to -\infty$ and $W[G(x)]\to b$ as $x\to\infty$

Let $r(t)$ be the pdf of a random variable $T\in\left[a,b\right]$ for $-\infty\le a<b<\infty$ and $W\left[G(x)\right]$ be a function of the cdf of a random variable X .

Then, the pdf corresponding to equation (3) is given by;

$$f(x)=\left\{\frac{d}{dx}W\left[G(x)\right]\right\}r\{W\left[G(x)\right]\} \tag{5}$$

Proposition 1: Let X be any arbitrary random variable with cdf $G(x;\xi)$ and pdf $g(x;\xi)$. Also, let $T\in(a,b)$ be a random variable with a pdf $r(t)$. Furthermore, let our proposed link function with cdf and pdf be given as:

$$H(x;\alpha,\beta,\psi)=\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta} \tag{6}$$

and

$$h(x;\alpha,\beta,\psi)=\frac{\alpha\beta g(x,\psi)}{[1-G(x;\psi)]^{2}}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta-1} \tag{7}$$

Then, the cdf of NOGEE-G family of distributions is given as

$$F(x;\alpha,\beta,\theta,\psi)=1-\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta} \tag{8}$$

Proof: Using the Lehmman II as the Transformed (T) and odd generalized exponential-G family as the Transformer (X), we have

$$F(x;\theta,\xi)=\int_{0}^{k}\theta h(t;\xi)\left[1-H(t;\xi)\right]^{\theta-1}\partial t$$

Where $k=\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}$

Let $y=\left[1-H(t;\xi)\right]$

When $t=0$, $y=1$ and when $t=k$, $y=1-k$

$$\frac{\partial y}{\partial t}=-h(t;\xi)\Rightarrow \partial t=\frac{\partial y}{-h(t;\xi)}$$

$$F(x;\theta,\xi)=\theta\int_{1-k}^{1}h(t;\xi)\left[y\right]^{\theta-1}\frac{\partial y}{h(t;\xi)}$$

$$F(x;\theta,\xi)=\theta\int_{1-k}^{1}h(t;\xi)\left[y\right]^{\theta-1}\frac{\partial y}{h(t;\xi)}$$

$$F(x;\theta,\xi)=\int_{1-k}^{1}\left[y\right]^{\theta-1}\partial y$$

$$F(x;\theta,\xi)=\theta\left[\frac{y^{\theta}}{\theta}\right]_{1-k}^{1}$$

$$F(x;\theta,\xi)=y^{\theta}\Big|_{1-k}^{1}$$

$$F(x;\theta,\xi)=1-\left[1-k\right]^{\theta}$$

$$F(x;\alpha,\beta,\theta,\psi)=1-\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta}$$

That completes the proof.

To get the pdf of the NOGEE-G family of distribution, equation (8) is differentiated with respect to x as

$$\frac{\partial F(x;\alpha,\beta,\theta,\psi)}{\partial x}=\frac{\alpha\beta\theta g(x,\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta-1}\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta-1}$$

$$f(x;\alpha,\beta,\theta,\psi)=\frac{\alpha\beta\theta g(x,\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta-1}\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta-1} \quad (9)$$

It suffices to show that

$$\int_0^{\infty} f(x;\alpha,\beta,\theta,\psi)=1$$

$$\int_0^{\infty}\frac{\alpha\beta\theta g(x,\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta-1}\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta-1}\partial x$$

Let $m=1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}$

when $x=0, m=0$ and when $x=\infty, m=1$

$$\frac{\partial m}{\partial x}=\frac{\alpha g(x;\psi)}{\left[1-G(x;\psi)\right]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}$$

$$\alpha\beta\theta\int_0^1\frac{g(x,\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}[m]^{\beta-1}\left[1-[m]^{\beta}\right]^{\theta-1}\frac{\partial m}{\frac{\alpha g(x;\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}}$$

$$\beta\theta\int_0^{\infty}[m]^{\beta-1}\left[1-[m]^{\beta}\right]^{\theta-1}\partial m$$

Let $z=1-[m]^{\beta}$

when $m=0, z=1$ and when $m=1, z=0$

$$\frac{\partial z}{\partial m}=-\beta m^{\beta-1}$$

$$\beta\theta\int_0^1[m]^{\beta-1}[z]^{\theta-1}\partial m$$

$$\beta\theta\int_0^1[m]^{\beta-1}[z]^{\theta-1}\frac{\partial z}{\beta m^{\beta-1}}$$

$$\theta\int_0^1[z]^{\theta-1}\partial z$$

$$\theta\left[\frac{z^{\theta}}{\theta}\right]_0^1$$

$$z^{\theta}\Big|_0^1$$

$$1-0$$

$$1$$

Therefore,

$$f(x;\alpha,\beta,\theta,\psi)=\frac{\alpha\beta\theta g(x,\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta-1}\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta-1}$$ is a valid pdf.

3. Statistical properties

3.1 Survival function

The survival function of the NOGEE- family of distributions is given as

$$S(x;\alpha,\beta,\theta,\psi)=1-\left\{1-\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta}\right\} \quad (10)$$

3.2. Hazard rate function

The hazard rate function is an important measure used to characterize a life phenomenon. Hence, the hazard rate function of the NOGEE- family of distributions is given as:

$$\tau(x;\alpha,\beta,\theta,\psi)=\frac{\frac{\alpha\beta\theta g(x,\psi)}{[1-G(x;\psi)]^2}e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta-1}\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta-1}}{1-\left\{1-\left[1-\left[1-e^{-\alpha\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)}\right]^{\beta}\right]^{\theta}\right\}} \quad (11)$$

3.3.Quantile Function

The quantile function of the NOGEE- family is easily simulated by inverting the cdf (8) as follows: if q has a uniform $U(0,1)$ distribution, then the solution of the nonlinear equation is given by

$$x_u=F^{-1}=\left\{1-\left[1-\frac{1}{\alpha}\log\left(\frac{\left(1-u^{1/\theta}\right)}{1-\left(1-u^{1/\theta}\right)}\right)^{1/\beta}\right]\right\} \quad (12)$$

The median of the NOGEE- family of distributions can be obtained by setting Q = 0.5 in (12) as:

$$Med=F^{-1}=\left\{1-\left[1-\frac{1}{\alpha}\log\left(\frac{\left(1-0.5^{1/\theta}\right)}{1-\left(1-0.5^{1/\theta}\right)}\right)^{1/\beta}\right]\right\} \quad (13)$$

4. Sub-models of the NOGEE- family of distributions

In this section, we provide some sub-models of the NOGEE- family. The cdf and pdf of the family of distributions were presented as follows. However, the two sub-models of this family of distributions with corresponding baseline of the Frechet (Fr) and Gompertz (Go) distributions show the new family's flexibility.

4.1 The NOGEE-Fréchet (NOGEE-Fr) distribution

Lemma1: The NOGEE-Fr distribution is defined from (8) by taking $G(x)=e^{-x^{-\sigma}}$ and $g(x)=\sigma x^{-\sigma-1}e^{-x^{-\sigma}}$ as the cdf and pdf of Fréchet distribution with parameters σ .

The cdf and pdf of the (NOGEE-Fr) distribution are given as

$$F(x;\alpha,\beta,\theta,\sigma)=1-\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta} \quad (14)$$

and

$$f(x;\alpha,\beta,\theta,\sigma)=\frac{\alpha\beta\theta\sigma x^{-(\sigma+1)}e^{-x^{-\sigma}}}{\left[1-e^{-x^{-\sigma}}\right]^2}e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta-1}\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta-1} \quad (15)$$

where $\alpha>0$ is scale parameter and $\beta>0,\ \theta>0$ and $\sigma>0$ are shape parameters.

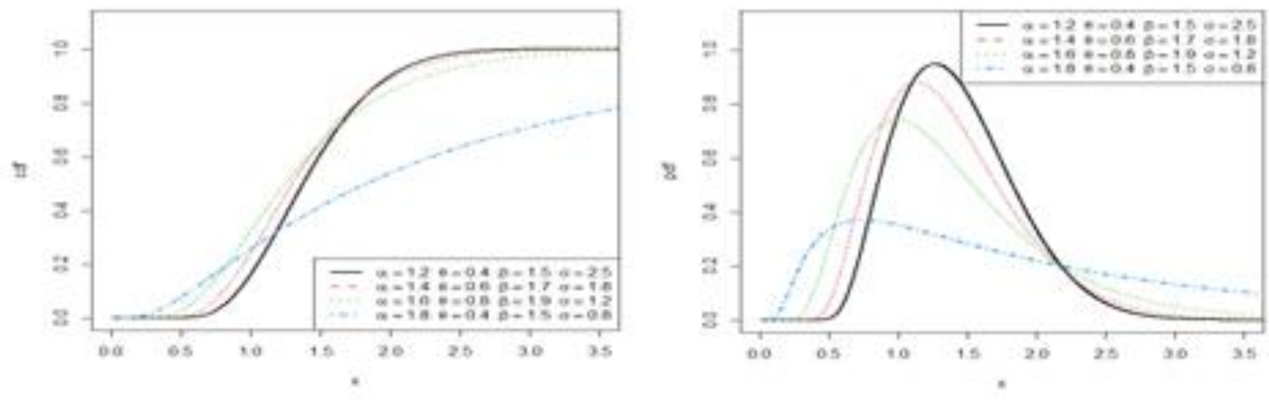

Figure 1: Plots of cdf and pdf of the NOGEE-Fr distribution for different parameter values.

4.1.1 The Survival function of NOGEE-Fr distribution is given by

$$S(x;\alpha,\beta,\theta,\sigma)=1-\left[1-\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta}\right] \quad (16)$$

4.1.2 The hazard rate function of NOGEE – Fr distribution is given by

$$\tau(x;\alpha,\beta,\theta,\sigma)=\frac{\alpha\beta\theta\sigma x^{-(\sigma+1)}e^{-x^{-\sigma}}e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta-1}\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta-1}}{\left[1-e^{-x^{-\sigma}}\right]^2 1-\left[1-\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta}\right]} \quad (17)$$

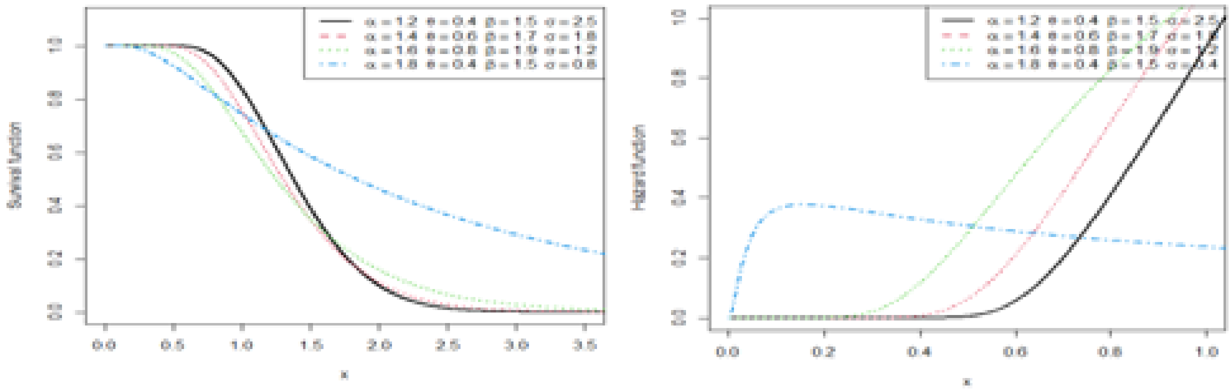

Figure 2: Plots of the survival function and hazard function of the NOGEE-Fr distribution for different parameter values.

4.2 The NOGEE-Gompertz (NOGEE-Go) distribution

Lemma 2: The NOGEE-G distribution is defined from (8) by taking $G(x)=e^{-\sigma(e^x-1)}$ and

$g(x;\psi)=\sigma e^x e^{-\sigma(e^x-1)}$ as the cdf and pdf of Gompertz distribution with parameters σ.

Hence, the cdf and pdf of the (NOGEE-Go) distribution are given as

$$F(x;\alpha,\beta,\theta,\sigma)=1-\left\{1-\left[1-e^{-\alpha\left(\frac{1-e^{-\sigma(e^x-1)}}{e^{-\sigma(e^x-1)}}\right)}\right]^{\beta}\right\}^{\theta} \qquad (18)$$

and

$$f(x;\alpha,\beta,\theta,\sigma)=\frac{\alpha\beta\theta\sigma e^x}{e^{-\sigma(e^x-1)}}e^{-\alpha\left(\frac{1-e^{-\sigma(e^x-1)}}{e^{-\sigma(e^x-1)}}\right)}\left[1-e^{-\alpha\left(\frac{1-e^{-\sigma(e^x-1)}}{e^{-\sigma(e^x-1)}}\right)}\right]^{\beta-1}\left\{1-\left[1-e^{-\alpha\left(\frac{1-e^{-\sigma(e^x-1)}}{e^{-\sigma(e^x-1)}}\right)}\right]^{\beta}\right\}^{\theta-1} \qquad (19)$$

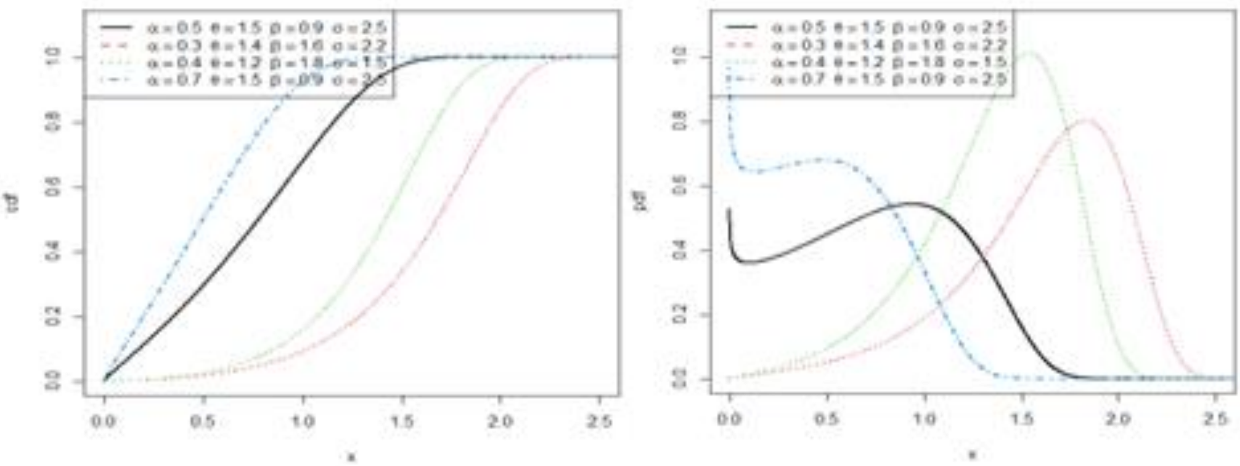

Figure 3: Plots of cdf and pdf of the NOGEE-Go distribution for different parameter values.

4.2.1The Survival function of NOGEE-Go distribution is given by

$$S(x;\alpha,\beta,\theta,\sigma) = 1 - \left[1 - \left\{1 - \left[1 - e^{-\alpha\left(\frac{1-e^{-\sigma(e^{x}-1)}}{e^{-\sigma(e^{x}-1)}}\right)}\right]^{\beta}\right\}^{\theta}\right] \quad (20)$$

4.2.2 The hazard rate function of NOGEE – Go distribution is given by

$$\tau(x;\alpha,\beta,\theta,\sigma) = \frac{\alpha\beta\theta\sigma x^{-(\sigma+1)} e^{-x^{-\sigma}} e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta-1}\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta-1}}{\left[1-e^{-x^{-\sigma}}\right]^{2} 1-\left[1-\left\{1-\left[1-e^{-\alpha\left(\frac{e^{-x^{-\sigma}}}{1-e^{-x^{-\sigma}}}\right)}\right]^{\beta}\right\}^{\theta}\right]} \quad (21)$$

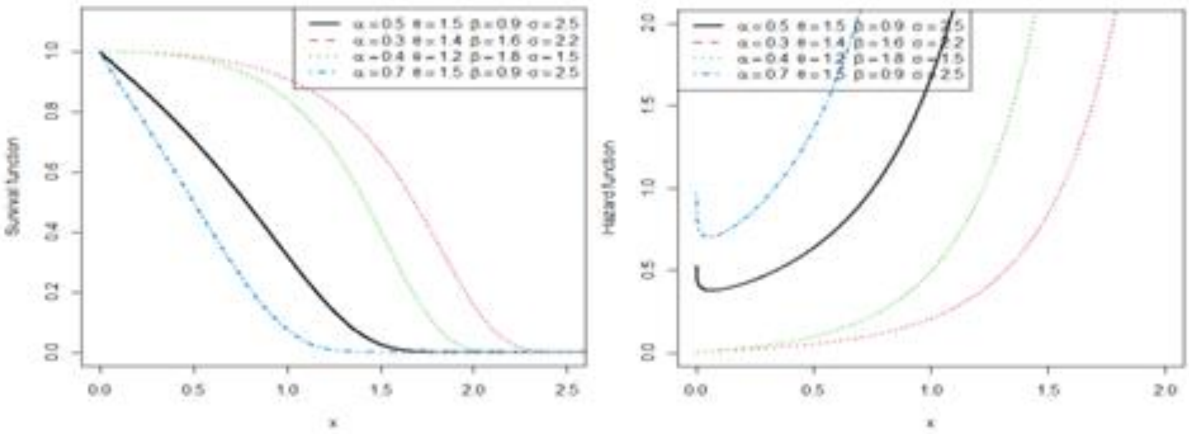

Figure 4: Plots of the survival function and hazard function of the NOGEE-Go distribution for different parameter values.

5 Parameter Estimation

This section will estimate the derivation of unknown parameters of the NOGEE – family of distributions using the Maximum Likelihood Estimators.

Let $X_1, X_2, \ldots, X_n$ to be a random sample of size n from NOGEE – (ξ), where $(\alpha,\beta,\theta,\xi)$, is the parameter function L of the probability density function given as:

$$L(\xi) = \prod_{i=1}^{n} f(x;\alpha,\beta,\theta,\xi).$$

Then, the log-likelihood function is given as:

$$L(\xi) = \frac{n\log\alpha + n\log\beta + n\log\theta}{2\sum_{i=1}^{n}\log[1-G(x;\xi)]} + \sum_{i=1}^{n}\log[g(x;\xi)] - \alpha \sum_{i=1}^{n\sum\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]} \log$$

$$+(\beta-1)\sum_{i=1}^{n}\log\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right] + (\theta-1)\left\{1-\left[1-e^{-\alpha\left(\frac{e^{\frac{\lambda}{\sigma}\left(e^{\sigma x}-1\right)}}{1-e^{\frac{\lambda}{\sigma}\left(e^{\sigma x}-1\right)}}\right)}\right]^{\beta}\right\} \quad (22)$$

To obtain the maximum likelihood estimates of the parameters $\alpha,\beta,\theta,$, we partially differentiate (22) with respect to the parameters α,β,θ and ξ . This gives:

$$\frac{\partial L}{\partial\theta} = \frac{n}{\theta} + (\theta-1)\sum_{i=1}^{n}\log\left\{1-\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta-1}\right\} = 0 \quad (23)$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}+(\beta-1)\sum_{i=1}^{n}\log\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]+(\theta-1)\sum_{i=1}^{n}\frac{\left\{1-\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta-1}\right\}\log\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]}{1-\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta}}=0 \quad (24)$$

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\log\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)-(\beta-1)\sum_{i=1}^{n}\left[\frac{\left[e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]\log\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}{\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]}\right]$$

$$-(\theta-1)\sum_{i=1}^{n}\left[\frac{\beta\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta-1}\left[e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]\log\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]}{1-\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta}}\right]=0 \quad (25)$$

$$\frac{\partial L}{\partial \xi}=\sum_{i=1}^{n}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)-\alpha\sum_{i=1}^{n}\log\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)-(\beta-1)\sum_{i=1}^{n}\left[\frac{\beta\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta-1}e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}}{\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]}\right]-(\theta-1)\sum_{i=1}^{n}\left[\frac{\beta\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}}{1-\left[1-e^{-\alpha\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)}\right]^{\beta}}\right]=0 \quad (26)$$

Since the solution to the above non-linear equations (23) - (26) cannot be solved analytically, a numerical solution is applied by the Newton-Raphon technique.

5.1 Applications to real-life data sets

This section will fit the NOGEE-Fr distribution to two real-life data sets to demonstrate its applicability and flexibility. The goodness of fit of NOGEE-Fr distribution would be compared with five models comprising the baseline distribution, namely, Kumaraswamy Fr (KFr) distribution, exponentiated Fr (EFr) distribution, Transmuted Fr (TFr), two parameters Fr (2Fr) distribution and one parameter Fr (1Fr) distribution. The model comparison would be based on the minimized log-likelihood estimate and the following information statistics: Akaike information criterion (AIC) and Bayesian information criterion (BIC). The model with the smallest minimized log-likelihood and information statistics value is the best.

Data set 1: represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are given as: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 1: The models' MLEs and performance requirements based on data set 1

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	ll	AIC	BIC
NOGEE-Fr	2.2615	109.7045	19.1686	0.1463	-94.4668	196.9337	206.0403
KFr	1.6209	0.4309	1.9751	0.0966	-95.3479	198.6959	207.8025
EFr	54.4861	-	0.4785	167.5938	-105.6273	217.2547	224.0847
TFr	0.6830	-0.9110	1.3157	-	-112.6363	231.2725	233.8259
2Fr	1.0585	1.1730	-	-	-118.1660	240.3320	244.8854
1Fr	-	-	-	1.1871	-118.3069	238.6138	240.8905

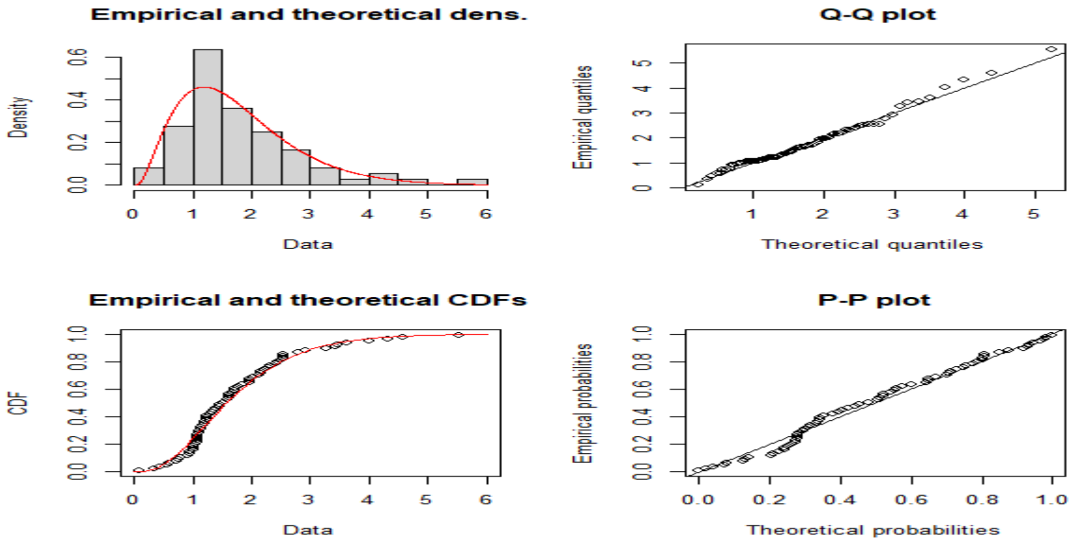

Figure 5: Histogram and empirical density plots for data set 1

Data set 2: was given by Lee (1992), and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938.: The data set is as follows:
0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 1.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Table 2: The models' MLEs and performance requirements based on data set 2

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	ll	AIC	BIC
NOGEEFr	0.0363	0.1184	0.4986	1.3668	-579.2023	1166.4050	1177.5880
KFr	0.0003	0.1654	0.4824	0.0015	-833.4187	1674.8370	1686.0210
EFr	549.4801	-	0.3725	13.0551	-599.4801	1204.9600	1213.3480
TFr	1.1629	-1.4265	0.4419	-	-660.4618	1326.9240	1330.5150
2Fr	16.8999	0.6523	-	-	-636.7940	1277.5880	1283.1800
1Fr	-	-	-	0.3836	-727.5931	1457.1860	1459.9820

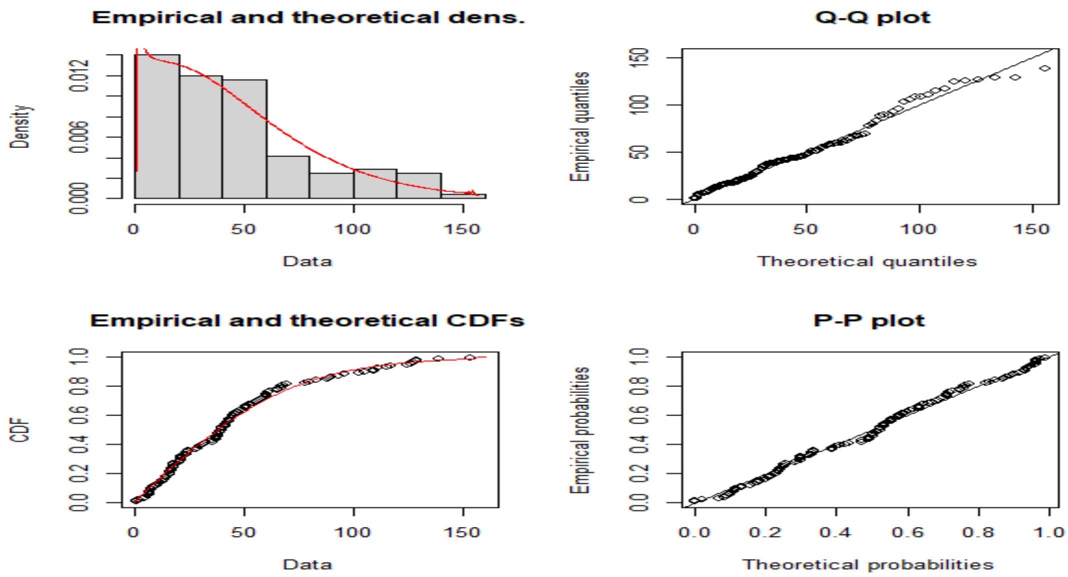

Figure 6: Histogram and empirical density plots for data set 2

5.3 Results

NOGEE-Fr and NOGEE-Go distributions are the two sub-models proposed in this study. The MLEs and the performance of the parameters are calculated. For the distributions, NOGEE-Fr distribution is used to test the goodness-of-fit of the proposed new family. The model with smaller values of these statistics is thought to be the best model for comparison. Table 1 and Table 2 provide the estimated values and the results for each parameter of the NOGEE-Fr distribution and that of its sub-models. According to Tables 1 and 2, the NOGEE-Fr distribution fits the data better compared to other distributions. As a result, the NOGEE-Fr distribution best fits this data and is a very strong competitor to other distributions.

CONCLUSION

In this paper, we proposed a new family of distributions, called the New Odd Generalized Exponentiated Exponential-G family of distributions. We use two applications on a set of real-life data to compare the performance of the NOGEE-Fr with other existing distributions. The analysis results showed that the NOGEE-Fr distribution is the best distribution for modeling these type of data sets compared to other distributions used in this paper.

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