

ORIGINAL RESEARCH ARTICLE

Solution of Time Independent System of First Order Ordinary Differential Equations Using Embedding Method in Fuzzy Environment.

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ABSTRACT

In this study, time independent system of first order fuzzy ordinary differential equations is solved using embedding method. Results for the system of FODEs are obtained and a case of non-homogeneous system of FODEs is established and used to find the Eigen values in an $n \times n$ matrix with negative real numbers \mathfrak{R}^- consisting of n fuzzy numbers. Examples are constructed using the principles of existence and uniqueness. It is recommended that further study should look at the possibilities of generalizing this work to n^{th} order system of first order ordinary differential equations using embedding method in fuzzy environment.

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INTRODUCTION

The study of fuzzy differential equations (FDEs) has been extensively developed in the past few years. As one of the research field in differential equations, FDE is very important topic from the theoretical point of view as well as areas of applications such as Physics, Economics, Engineering and Bio-Mathematics. The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1972). Later, Dubois and Prade (1982) presented a concept of the fuzzy derivative based on the extension principle. Buckley and Feuring (2000) compared various derivatives of fuzzy function that have been presented in the various literatures. Recently, Bede and Gal (2004) introduced a concept for strongly generalized differentiability of fuzzy functions. Allahviranloo et al. (2009) used the concept of generalised differentiability and applied differential transformation methods for solving fuzzy differential equations, Khastan, et al. (2011) studied first order linear fuzzy differential equations by using the generalized differentiability concept.

The first order linear ODE $\dot{X}(t) = \alpha(A - I_n)[t(A - I_n) + I_n]^{-1}$ in fuzzy environment, with initial condition $X(0) = X_0$ where A is real $n \times n$ matrix, X_0 is a vector consisting of some fuzzy numbers will be solved using embedding method. Sadeghi, et al. (2011) work only considered time dependent system of linear FODEs without giving

attention to time independent system of the equations. Considering a case of time independent system will give better results and further broaden the scope of the work. Therefore, this study aims at addressing the aforementioned problem and also considers extending it to include time independent system of ODE with the initial condition $X(0) = X_0$ and use the embedding method to solve the ODE in fuzzy environment. Finally, existence and uniqueness of solution to the FODEs is also considered. The initial value problem

$$\dot{X}(t) = \alpha(A - I_n)[t(A - I_n) + I_n]^{-1} X(t), X(0) = X_0 \tag{1}$$

where A is an $n \times n$ matrix that has non-negative eigenvalues in real axis was considered by Sadeghi et al. (2011). The initial condition of equation (1) was fuzzified and the embedding method was used to solve the fuzzified initial value problem and the necessary and sufficient condition for the existence of fuzzy solution $X(t)$ was established. The general form of FDE due to Sadeghi et al. (2011) is in the form

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In other words

$$s_1, s_2, A = s_1 - s_2, s_1 + s_2 = |A| = (|a_{ij}|),$$

$$i, j = 1, \dots, n \text{ and}$$

$$f_0(t)x^n(t) \oplus f_1(t)x^{n-1}(t) \oplus f_2(t)x^{n-2}(t) \oplus \dots \oplus x'(t)f_{n-1}(t) \oplus x(t)f_n(t) = G(t)$$

(2)

where

$$f_i(t), i = 0, 1, \dots, n$$

(3)

are finite polynomials that contain no common factor with a fuzzy initial and boundary value below.

$$x(t_0) = k_1, \dots, x^{(n-1)}(t_0) = k_n$$

(4)

$$x'(t) = k_i, i = 1, \dots, n.$$

(5)

Equation (2) can either be linear homogeneous, linear nonhomogeneous, nonlinear homogeneous or nonlinear nonhomogeneous ODEs, which can be considered in fuzzy environment. Equation (2) will be solved subject to equation (4)

METHOD

Based on the work of [Sadeghi et al., \(2011\)](#), we therefore, considered extending the Embedding method to solve system of ODE in fuzzy environment. Consider the following equation

$$\sum_{j=1}^n a_{ij} x_j = y_i, i = 1, \dots, n$$

(1)

where the coefficient matrix

$A = (a_{ij}), 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y_i \in E, 1 \leq i \leq n$ (where E is the family of all fuzzy number) is a fuzzy system of equation. Two crisps $n \times n$ systems for all i can be extended to an $3n \times 3n$ crisp system in the following

(i) if $a_{ij} \geq 0$ for $i, j = 1, \dots, 3n$, then

$$s_{ij} = s_{i+n, j+n} = a_{ij},$$

(ii) if $a_{ij} < 0$ for $i, j = 1, \dots, 3n$, then

$$s_{ij} = s_{i+n, j+n} = a_{ij}.$$

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \bar{X} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}, \underline{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \bar{Y} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_n \end{pmatrix}.$$

The following lemmas are considered to be important with regard to solutions of ODEs using embedding method in fuzzy environment.

Lemma 1: ([Sadeghi et al., 2011](#)): The unique solution X to equation (3.42) is a fuzzy vector for arbitrary Y if and only if S^{-1} is nonnegative, i.e., $(s_{ij})^{-1} \geq 0$.

Lemma 2: ([Sadeghi et al., 2011](#)): The matrix S is nonsingular if and only if the matrices $s_1 - s_2$ and $s_1 + s_2$ are both singular. For proofs of lemmas 3.7 and 3.8, refer to [Sadeghi et al. \(2011\)](#).

Lemma 3: (i and ii differentiability concept)

if y' and y'' are (i) differentiable,

$$y'(t) = (\underline{y}'(t, \alpha), \bar{y}'(t, \alpha)) \text{ and}$$

$$y''(t) = (\underline{y}''(t, \alpha), \bar{y}''(t, \alpha))$$

if y' is (i) differentiable and y'' (ii) differentiable,

$$y'(t) = (\underline{y}'(t, \alpha), \bar{y}'(t, \alpha))$$

$$y''(t) = (\underline{y}''(t, \alpha), \bar{y}''(t, \alpha))$$

if y' is (ii) differentiable and y'' is (i) differentiable,

$$y'(t) = (\underline{y}'(t, \alpha), \bar{y}'(t, \alpha)) \text{ and}$$

$$y''(t) = (\underline{y}''(t, \alpha), \bar{y}''(t, \alpha)).$$

if y' and y'' are (ii) differentiable

$$y'(t) = (\underline{y}'(t, \alpha), \bar{y}'(t, \alpha)) \text{ and}$$

$$y''(t) = (\underline{y}''(t, \alpha), \bar{y}''(t, \alpha)).$$

Lemma 4: ([King et al., 2003](#)): Suppose that f is continuous on a domain R of the (t, x) plane defined for $a, b > 0$, by $R = \{(t, x) : |t - t_0| \leq a, \|x - x_0\| \leq b\}$, and that f is Lipschitz in x on R .

$$M = \sup_{(t,x) \in R} \|f(t,x)\| < \infty \text{ and } \alpha = \min\left(a, \frac{b}{M}\right)$$

Then the sequence defined by

$$\phi_0 = x_0, |t - t_0| \leq \alpha$$

$$\phi_i = x_0 + \int_{t_0}^t f(s, \phi_{i-1}(s)) ds \quad i \geq 1, |t - t_0| \leq \alpha$$

Converges uniformly on the interval $|t - t_0| \leq \alpha$ to ϕ

Lemma 5: (James 2000): Principles of Mathematical Induction.

Let $p(n)$ be a statement with the property that if

- (i) $p(1)$ is true
- (ii) whenever $p(k)$ is true, then $p(k+1)$ is true, then $p(n)$ is true for every positive integer n

Lemma 6: (James 2000):

If a sequence of function $\{\phi_k(t)\}$ converges uniformly and that the $\phi_k(t)$ are continuous on the interval $|t - t_0| \leq \alpha$, then $\lim_{n \rightarrow \infty} \int_{t_0}^t \phi_n(s) ds = \int_{t_0}^t \lim_{n \rightarrow \infty} \phi_n(s) ds$ and that $\phi(t) = \lim_{n \rightarrow \infty} \phi_n(t)$

Method of successive approximation (King, et al. (2003)).

The following method of successive approximation is useful in establishing the existence of solution of FODEs considered in this study.

$$\left. \begin{aligned} y_0(t) &= y_0 \\ y_1(t) &= y_0 + \int_{t_0}^t f(y_0, s) ds \\ y_2(t) &= y_0 + \int_{t_0}^t f(y_1(s), s) ds \\ &\vdots \\ y_{k+1}(t) &= y_0 + \int_{t_0}^t f(y_k(s), s) ds \end{aligned} \right\}$$

Equation (1) will be modified and use to establish the existence of solution for the third order equations. Prior to modification of equation (1), we must show that the elements of the sequence are well defined. Thus consider the following results;

Lemma 6: (King et al., 2003): If $\alpha = \min\left(a, \frac{b}{m}\right)$, then the successive approximations

$$y_0(t) = y_0,$$

$$y_{k+1}(t) = y_0 + \int_{t_0}^t f(y_k(s), s) ds$$

are well defined in the interval $I = \{t \mid |t - t_0| \leq \alpha\}$, and on this interval

$$|y_k(t) - y_0| < M |t - t_0| \leq b$$

where $|f| < M$.

Results

Consider the initial value problem

$$\begin{aligned} \dot{X}(t) &= \alpha(A - I_n)[t(A - I_n) + I_n]^{-1}, \\ X(0) &= X_0 \end{aligned} \tag{2}$$

where A is $n \times n$ matrix that has nonnegative eigenvalues in real axis.

Since

$$M = \sup_{(t,x) \in R} \|f(t,x)\| < \infty \text{ and } \alpha = \min\left(a, \frac{b}{M}\right)$$

Then the sequence defined by

$$\phi_0 = x_0, |t - t_0| \leq \alpha$$

$$\phi_i = x_0 + \int_{t_0}^t f(s, \phi_{i-1}(s)) ds \quad i \geq 1, |t - t_0| \leq \alpha$$

Which follows from equation (2) that

$$\dot{X} = \alpha(A - I_n)[A - I_n + I_n]^{-1}, \tag{3}$$

$$\dot{X} = \alpha(A - I_n)A^{-1},$$

$$\dot{X} = \alpha(1 - I_n A^{-1}), X_0 = X(0).$$

For the non homogeneous case,

$$y_{k+1}(t) = y_0 + \int_{t_0}^t f(y_k(s), s) ds$$

then

$$\dot{X} = \alpha(1 - I_n A^{-1}) + f(y), X_0 = X(0) \tag{4}$$

where $f(y)$ is the inhomogeneous term. Equation (4) has a unique solution

$$\dot{X} = A^\alpha x_0 + \int_0^k SA^{-2\alpha} f(y) dx, X_0 = X(0). \tag{5}$$

Suppose:

$$(i) \int_0^y (A - I_n)[SA - SI_n + I_n]^{-1} dx = \log A$$

(ii) $\log A$ commutes with A^{-1} .

Considering equation (5) and multiply both sides by the factor $e^{-\alpha \log A}$ we have

$$\left[\dot{X} - \alpha(1 - I_n A^{-1})X(y) \right] e^{-\alpha \log A} = f(y) e^{-\alpha \log A}. \tag{6}$$

Therefore,

$$\frac{d}{dx} \left(e^{-\alpha \log A} X(y) \right) = f(y) e^{-\alpha \log A} \tag{7}$$

$$\int_0^s \frac{d}{dx} e^{-\alpha \log A} X(y) dy = \int_0^s f(y) e^{-\alpha \log A} dy. \tag{8}$$

$$|y_k(t) - y_0| < M |t - t_0| \leq b$$

where $|f| < M$, then it implies that

$$e^{-\alpha \log[S(A - I_n) + I_n]} X(s) - s(0) = \int_0^s f(y) e^{-\alpha \log A} dy \tag{9}$$

$$X(s) = e^{\alpha \log[S(A - I_n) + I_n]} X(0) + \int_0^s f(y) e^{\alpha \log[S(A - I_n) + I_n]} e^{-\alpha \log A} dy. \tag{10}$$

Therefore,

$$X = A^\alpha X_0 + \int_0^k SA^{-2\alpha} f(y) dx. \tag{11}$$

Equation (11) indicated that ODE (2) is extended to time independent integral equation.

Constructed Example and Results Validation

Existence of Solution to System of First Order FODEs

Results will be tested and validated by considering the work of Sadeghi et al., 2011 based on the existence and uniqueness theorems presented in this study. Consider the following equations

$$Y(t) = Y^0 + \int_{t_0}^t \alpha (A - I_n) [p(s)(A - I_n) + I_n]^{-1} ds, \tag{12}$$

$$X(t) = X^0 + \int_{t_0}^t \alpha (A - I_n) [K(s)(A - I_n) + I_n]^{-1} ds. \tag{13}$$

Consider the sequence of iterates

$$\begin{aligned} y^0(t) &= y^0 \\ y'(t) &= y^0 + \int_{t_0}^t f(s, y(s), y'(s)) ds \\ &\vdots \\ y^{N+1}(t) &= y^0 + \int_{t_0}^t f(s, y(s), y^N(s)) ds \end{aligned} \tag{14}$$

Each iteration stays within the domain of the definition of $f(t, Y)$, which is clearly necessary if the next iterates is to be defined. We show that the iterates are defined for $|t - t_0| \leq \delta$ and satisfy the inequalities

$$|X^{N+1}(t) - X^0| \leq \beta.$$

First it is trivial that

$$|y^0(t) - y^0| = 0 \leq \beta, \quad |X^n(t) - X^0| = 0 \leq \beta \quad \text{for } |t - t_0| \leq \delta, n = 0, 1, \dots, N.$$

From equations (14)

$$\begin{aligned}
 |X^{N+1}(t) - X^0| &\leq \left| \int_{t_0}^t \alpha(A - I_n)[P(s)(A - I_n) \right. \\
 &\quad \left. + I_n]ds \right| \\
 &\leq M |t - t_0| \leq \beta, \text{ for } |t - t_0| \leq \delta
 \end{aligned}
 \tag{15}$$

To show that the sequence of iterates is uniformly and absolutely convergent. From equation (15) it follows that

$$\begin{aligned}
 X^{N+1}(t) - X^N(t) &= \int_{t_0}^t \alpha(A - I_n)[p^N(s)(A - I_n)^N + \\
 &\quad I_n]^{-1} - \alpha(A - I_n)[K^{N+1}(s)(A - I_n)^{N+1} + \\
 &\quad I_n]^{-1} ds
 \end{aligned}
 \tag{16}$$

Taking the norm of both side of equation (16) and using Lipschitz condition we have

$$\begin{aligned}
 \|X^{N+1}(t) - X^N(t)\| &\leq L \left\| \int_{t_0}^t \|p^N(s) - k^{N+1}(s)\| ds \right\| \\
 \|X'(t) - X^0(t)\| &\leq M |t - t_0| \quad X^0(t) = X_0.
 \end{aligned}$$

Hence, by the comparison test, $X^N(t)$ converges uniformly for $|t - t_0| \leq \delta$ to a limit function $X(t)$.

$$\begin{aligned}
 &\left\| \int_{t_0}^t \alpha(A - I_n)[p^N(s)(A - I_n)^N + I_n]^{-1} \right. \\
 &\quad \left. - \alpha(A - I_n)[K^{N+1}(s)(A - I_n)^{N+1} \right. \\
 &\quad \left. + I_n]^{-1} ds \right\| \\
 &\leq \left\| \int_{t_0}^t \|p^N(s) - K^{N+1}(s)\| ds \right\| \\
 &\leq L\delta \max_{|t-t_0| \leq \delta} \|X(t) - X^N(t)\|
 \end{aligned}$$

which tends to zero as $N \rightarrow \infty$. As the integrand $f(t, Y)$ is a continuous function, $y(t)$ is differentiable with respect to t , and, so $y(t)$ is a solution of the first order fuzzy differential equations (2).

Uniqueness of Solution to System of First Order FODEs

Let $X(t)$ and $Y(t)$ be solutions to equation (1) for $|t - t_0| \leq \delta$ such that

$$Y(t_0) = Y^0, X(t_0) = X^0, \text{ where } X^0, Y^0 \in D, \text{ it follows that}$$

$$\begin{aligned}
 Y(t) &= Y^0 + \int_{t_0}^t \alpha(A - I_n)[p(s)(A - I_n) + I_n]^{-1} ds \\
 &\quad ,
 \end{aligned}
 \tag{17}$$

$$\begin{aligned}
 X(t) &= X^0 + \int_{t_0}^t \alpha(A - I_n)[K(s)(A - I_n) + I_n]^{-1} ds.
 \end{aligned}
 \tag{18}$$

Subtracting equation (17) and (18) we have

$$\begin{aligned}
 Y(t) - X(t) &= Y^0 - X^0 + \int_{t_0}^t [p(s)(A - I_n) + \\
 &\quad I_n]^{-1} - [K(s)(A - I_n) + I_n]^{-1} ds
 \end{aligned}
 \tag{19}$$

$$\begin{aligned}
 0 \leq |Y(t) - X(t)| &\leq |Y^0 - X^0| + \left| \int_{t_0}^t L |p(s)^{-1} - \right. \\
 &\quad \left. K(s)^{-1}| ds \right|.
 \end{aligned}
 \tag{20}$$

Applying Gronwalls, with $r(t) = |Y(t) - X(t)|$, $\varepsilon = |Y^0 - X^0|$ and $\delta = L$. Hence the results followed by letting $Y^0 = X^0$, we see that $Y(t) = X(t)$ for all $|t - t_0| \leq \delta$. This established the uniqueness to equation (20)

DISCUSSION

The results for the time independent system of first order fuzzy ordinary differential equations were obtained in this study which we presented in equation (11) which determined the Eigenvalues in an $n \times n$ matrix for negative real numbers consisting of n fuzzy numbers.

We also saw the need to established the existence of solution to equation (2) which was obtained by utilizing lemmas (3) and (5) which yielded the results presented in Equation (16). While establishing the uniqueness of solution for the system of equation (2), we applied method of successive approximation and lemmas (4) and (5) which yielded the results presented in equation (19) and (20) respectively and hence the results follows.

CONCLUSION

The problem associated with non-inclusiveness of time independent system of FODEs to an existing work in the literature was raised earlier on. To tackle that and other related issues concerning this study, we employed embedding method to solve the time independent system of first order FODE based on gH-differentiability concept and the Eigenvalues for the given matrix were also obtained. Examples were presented. It is recommended that further study should look at the possibilities of generalizing this work to n^{th} order system of first order ordinary differential equations using embedding method in fuzzy environment.

CONTRIBUTION TO KNOWLEDGE

- i. Formation of time independent system of first order fuzzy ordinary differential equations
- ii. Employment of embedding method in solving the time independent system of first order ordinary differential equation

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