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ORIGINAL RESEARCH ARTICLE

Developing the Third Order Linear and Nonlinear Fuzzy Ordinary Differential Equations in Hilbert Space by the Extension Principle.

Hassan Habibu[1] and Tahir Alhaji[2].
[1]Department of Mathematics, College of Education, Waka-Biu, Borno State, Nigeria.
[2]Department of Mathematics, Modibbo Adama University of Technology, Yola, Adamawa State, Nigeria.

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ABSTRACT
In this study, a third nonlinear fuzzy ordinary differential equation is defined and established in Hilbert space using extension principle. The results obtained gives clear distinction between this work and the existing ones in the literature an example is formulated which shows that a closed subset of a Hilbert space is a Hilbert Space. It is recommended that future study should consider the development of system of third order linear and nonlinear fuzzy ordinary differential equations in Hilbert space by the extension principle.

INTRODUCTION

Fuzzy sets and fuzzy logic received significant impact on the evolution of many concepts with a strong relationships in various aspect of science. One of the principle contributions of Hilbert space and fuzzy logic is the generalization of the concepts and the relationships which may be based on crisp sets and fuzzy sets. Buckley and Feuring (2000).

A fuzzy differential equation is a differential equation in which some coefficients and parameters or boundary conditions are assumed to be a class of fuzzy sets. Puri and Ralescu (1983). Classes of fuzzy sets is mainly regarded as the class of fuzzy numbers consisting of sets that are fuzzy convex, normal, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. FDEs may be considered as a type of uncertain differential equations in which the uncertain values of parameters, coefficients, and boundary conditions are considered as fuzzy numbers. Seikkala (1987). Abu Arqub et al. (2016) proposed an approach based on the reproducing kernel Hilbert space method. Ahmadian et al. (2018) extended the fourth-order Runge–Kutta method to deal with first-order fuzzy differential equations. Discussions on different formulations of first-order linear fuzzy differential equations are given by Khastan and Rodríguez-López (2016). Gumah et al. (2020) used Gram-Schmidt orthogonalization to establish a system of orthogonal functions to obtain approximate-analytical solutions of FDEs. A study on first-order linear FDEs under differential inclusions is presented by Khastan and Rodríguez-López (2020).

The study of fractional differential equations of Riemann-Liouville and Caputo type in Hilbert spaces using exponentially weighted spaces of functions defined on The defined fractional operators by means of a functional calculus using the Fourier transform was considered. Main tools are extrapolation- and interpolation spaces. Main results obtained are the existence and uniqueness of solutions and the causality of solution operators for non-linear fractional differential equations. Kai, et al. (2019)

Differential equations in Hilbert spaces and applications to BVPs in non smooth domains was considered. The existence and uniqueness results in weighted Sobolev spaces for a solution of first ODE with operator coefficients of type $\frac{\partial u}{\partial t} - Au(t) = f(t)$ were obtained in an abstract version of elliptic boundary value problems in infinite cylinders. The results translate the classical ones to this setting. The difference between two solutions belonging to different weighted Sobolev spaces is a finite linear combination of singular functions depending on the eigenvalues of A and on the corresponding eigen spaces. They compute the coefficients of these singular functions by explicit y the Laurent series of the resolvent of A near an eigenvalue of the operator also they give an abstract polynomial resolution which corresponds to the resolution of an elliptic boundary value problem in an elliptic boundary value problem in an infinite cone with

Correspondence: Hassan Habibu. Department of Mathematics, College of Education, Waka-Biu, Borno State, Nigeria.
habibkantoma@gmail.com. Phone Number: +234 903 747 6579.
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polynomial right-hand sides. Finally, application to the abstract theory were presented. Serge (1989)

Hilbert space perspective on ordinary differential equations with memory term. was discussed and the ordinary differential equations with delay and memory terms in Hilbert spaces was also considered by introducing a time derivative as a normal operator in an appropriate Hilbert space, we develop a new approach to a solution theory covering integro-differential equations, neutral differential equations and general delay differential equations within a unified framework. They show that reasonable differential equations lead to causal solution operators. Anke, et al. (2012)

The solution of singular equations and linear equations in Hilbert space was considerd. The theorem on which this paper is based is an easy generalization of the fact that the nullspace of an operator is the orthocomplement of (the closure of) the range of its adjoint. Its significance, here, is the observation that this may be applied to give a computationally feasible algorithm for the problem of the title. Thomas (1975)

This study therefore is based on the second order linear homogeneous ODEs by the concept of generalized Hukuhara differentiability and using Extension Principle presented in Eljaoui et al. (2015)

Consider the second order differential equation

$$.\,y''(t) = f(t, y(t), y'),$$
$$y(0) = y_0 = (\underline{y}_0, \overline{y}_0),\ \ y'(0) = z_0 \ = (\underline{z}_0, \overline{z}_0)$$

Which is linear with respect to $(y(t), y'(t))$. Based on what has been observed in this study, Eljaoui et al. (2015) work was limited to the assumption that the fuzzy linear function

$$f(t, y(t), y'(t)) = a\,y(t) + b\,y'(t) + c(t)$$

is a crisp mapping. This research therefore, addresses those gaps identified above and as well formulated third order FODEs and established it in Hilbert space.

METHOD

Concept of Third Order FODEs in Hilbert Space by Extension Principle

Based on the works of Eljaoui et al., (2015), we therefore, formulated third order nonlinear ODE (1) with fuzzy initial conditions (2) .These results when obtained, will give a clear distinction between this work and the existing ones.

$$y'''(t) = ky(t) + y_c(t) - y(t)y'(t) \qquad (1)$$

where $y(t) = (\underline{y}(t,\alpha), \overline{y}(t,\alpha))$, with $y \neq y_c$, and k is constant with the following fuzzy initial condition

$$\begin{cases} y(0) = x_0 = (\underline{x}_0, \overline{x}_0) \\ y'(0) = y_0 = (\underline{y}_0, \overline{y}_0) \\ y''(0) = z_0 = (\underline{z}_0, \overline{z}_0) \end{cases} \qquad (2)$$

Lemma 1: Fuzzification of third order FODEs in Hilbert space (King et al., 2003): If f and the derivatives belong to $C^3(R)$, then the solution of equation (1) subject to the initial condition (2) is unique on the interval $|t - t_0| < \alpha$.

Lemma 2: extension principle : (John, 1990): Let X be a normed linear space (such as an inner product space), and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of elements of X .

(a) $\{f_n\}$ converges to $f \in X$, i.e, $f_n \to f$, if $\lim_{n\to\infty} \|f - f_n\| = 0$

i.e $\forall \varepsilon > 0, \exists N > 0$ such that $n > N \Rightarrow \|f - f_n\| < \varepsilon$

(b) $\{f_n\}$ is Cauchy if $\forall \varepsilon > 0, \exists N > 0$ such that $m, n > N \Rightarrow \|f_m - f_n\| < \varepsilon$

Lemma 3: (John, 1990): Euclidean space R with inner product space $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$,

$x = (x_1, x_2, \cdots, x_n) = (x_i)$, $y = y_i$ it is a Hilbert space as any finite dimensional normed space is a Banach space

Lemma 4: (John, 1990): Let X be an inner product space $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.

(1) Cauchy Schwarz inequality $|\langle x, y\rangle| \le \|x\|\|y\|$ with the inequality if and only if x and y are linearly dependent

(ii) $\langle \cdot, \cdot \rangle$ is continuous function from X to X to the corresponding scalar field

(iii) Parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ \forall\, x, y \in X$$

Parallelogram law is a sufficient and necessary condition for a norm $\|x\|$ to be the associated norm for an inner product on X

(iv) Polarization:

Retrieving the inner product from the norm satisfying parallelogram law

$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ for the real scalar field and

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2\right)$$

for complex field

Lemma 5: (John, 1990): Let H be a Hilbert space and M be a subspace of H. Then M is a Hilbert space (using the inner product from H) if and only if M is closed.

Lemma 6: (John, 1990): A closed subset of a Hilbert space is a Hilbert Space

Lemma 7: (John, 1990): Space

$l_2 := x = (x_1, x_2, \cdots, x_n, \cdots = (x_i) : \sum_{n=1}^{\infty} |x_n|^2 < \infty$. It is a Hilbert space with inner product $\langle x,y\rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n$

RESULTS

Consider the following third order differential equation (1). Using lemma 2 on equation (1) which is denoted by C^3 (the third Euclidean space). Again, applying the same lemma 2, with X as a subspace of C^3, then X is an inner product space $\langle \cdot,\cdot \rangle$ with the associated norm $\|\cdot\|$. Utilizing lemma 2, X will become a Hilbert space.

Suppose f is any limit point of X, such that $f_n \in X$ and $f_n \to f$. Then by using lemma 2, we see that $\{f_n\}$ is a convergent sequence in the space H and hence is said to be a Cauchy. Since each f_n belongs to X, then it is also Cauchy in X. Also, X is a Hilbert space, $\{f_n\}$ must therefore converges in X, i.e $f_n \to R$ for some $R \in X$. However, its limits are unique, so $f = R \in X$. Therefore X contains all its limits points and hence is closed.

Constructed Example

Suppose H is a Hilbert space and M is a subspace of H. Then M is said to be a Hilbert space if and only if M is closed. Which means that a closed subset of a Hilbert space is a Hilbert Space. Therefore, let space

$l_2 := x = (x_1, x_2, \cdots, x_n, \cdots = (x_i) : \sum_{n=1}^{\infty} |x_n|^2 < \infty$.

Hence it is a Hilbert space with inner product

$$\langle x,y\rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

Validation of the Result

At this point, the results obtained in this study will be validated by considering the work of Eljaoui et al., (2015):

1. Let $f(x)$ and $g(x)$ be continuous valued functions and c_1, c_2 be two real constants, then

$$L[c_1 f(x) + c_2 g(x)] = c_1 L[f(x)] + c_2 L[g(x)].$$

2. Let $f(x)$ be continuous valued function such that $e^{-px} f(x)$, $e^{-px} f'(x)$ and $e^{-px} f''(x)$ exist, are continuous, and Riemann integrable on $[0,\infty]$, the following are true.

(a) If $f(x)$ and $f'(x)$ are (i) gH -differentiable, then

$$L[f''(x)] = \{p^2 L[f(x)] - p f(0)\} - f'(0).$$

(b) If $f(x)$ is (i) gH -differentiable and $f'(x)$ is (ii) gH -differentiable, then

$$L[f''(x)] = \left(-f'(0)\right) - \{-p^2 L[f(x)] - \left(-p f(0)\right)\}$$

(c) If $f(x)$ is (ii) gH -differentiable and $f'(x)$ is (i) gH -differentiable, then

$$L[f''(x)] = \left(-p f(0)\right) - \left(-p^2 L[f(x)]\right) - f'(0)$$

(d) If $f(x)$ and $f'(x)$ are both (ii) gH -differentiables, then

$$L[f''(x)] = \left(-f'(0)\right) - p f(0) - p^2 L[f(x)].$$

Consider the following second order ODE in fuzzy environment presented by Eljaoui et al., (2015). We therefore develop the following results in Hilbert Space by utilizing Lemma 1 to Lemma 7 with the results obtained in this study

$$\begin{cases} y''(t) = f(t, y(t), y'), \\ y(0) = y_0 = (\underline{y}_0, \bar{y}_0) \in E \\ y'(0) = z_0 = (\underline{z}_0, \bar{z}_0) \in E \end{cases} \quad (1)$$

where $y(t) = (\underline{y}(t,\alpha), \bar{y}(t,\alpha))$ is a fuzzy valued function of $f \geq 0$ and $f(t, y(t), y'(t))$ is a fuzzy valued function, which is linear with respect to $y(t), y'(t)$. Applying Laplace transform to equation (1) we have

$$L[y''(t)] = L\left[f\left(t, y(t), y'(t)\right)\right]. \quad (2)$$

The process of solving equation (2) exists if y and y' are (i) gH -differentiable

$y'(t) = \left(\underline{y}'(t, \alpha), \bar{y}'(t, \alpha)\right)$ and $y''(t) = \left(\underline{y}''(t, \alpha), \bar{y}''(t, \alpha)\right)$,

$$L[y''(t)] = \{p^2 L[y(t)] - py(t)\} - y'(0). \quad (3)$$

Applying condition (2) on equation (3) yielded

$$L\left[f\left(t, y(t), y'(t)\right)\right] = \{p^2 L[y(t)] - py_0\} - z_0.$$

Using the α cut implies

$$L[\underline{f}(t, y(t), y'(t), \alpha)] = p^2 L[\underline{y}(t, \alpha)] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha) \quad (4)$$

$$L[\bar{f}(t, y(t), y'(t), \alpha)] = p^2 L[\bar{y}(t, \alpha)] - p\bar{y}_0(\alpha) - \bar{z}_0(\alpha). \quad (5)$$

where

$$L[\underline{f}(t, y(t), y'(t), \alpha)] = \left\{f(t, u, v), u \in \left(\underline{y}(t, \alpha), \bar{y}(t, \alpha)\right), v \in \left(\underline{y}'(t, \alpha), \bar{y}'(t, \alpha)\right)\right\}$$

and

$$L[\bar{f}(t, y(t), y'(t), \alpha)] = \left\{f(t, u, v), u \in \left(\underline{y}(t, \alpha), \bar{y}'(t, \alpha)\right), v \in \left(\underline{y}'(t, \alpha), \bar{y}(t, \alpha)\right)\right\}.$$

Therefore, the solution of equation (4) and (5) is obtained as

$$L[\underline{y}(t,\alpha)] = H_1(p,\alpha), \quad (6)$$

$$L[\bar{y}(t,\alpha)] = K_1(p,\alpha). \quad (7)$$

Consider the following second order differential equation (1). Using lemma 2 on equation (1) which is denoted by C^2 (the second Euclidean space). Again, applying the same lemma 2, with X as a subspace of C^2, then X is an inner product space $\langle \cdot, \cdot \rangle$ with the associated norm $\|\cdot\|$. Utilizing lemma 2, on $L[\underline{y}(t,\alpha)] = H_1(p,\alpha)$, $L[\bar{y}(t,\alpha)] = K_1(p,\alpha)$, we see that

X will become a Hilbert space. Therefore, to show that C^2 contains all its limits points,

Suppose f is any limit point of C^2, such that $f_n \in C^2$ and $f_n \to f$. Then by using lemma 2, 5 and 6. we see that $\{f_n\}$ is a convergent sequence in the space H and hence is said to be a Cauchy. Since each f_n belongs to C^2, then it is also Cauchy in C^2. Also, C^2 is a Hilbert space, $\{f_n\}$ must therefore converges in C^2, i.e $f_n \to R$ for some $R \in C^2$. Therefore C^2 contains all its limits points and hence is closed. This shows that the second order differential equation (1) presented by Eljaoui et al., (2015) is defined in Hilbert space based on the results obtained in this study and hence validate our results.

DISCUSSION

The third order nonlinear equation (1) was defined and developed in Hilbert space by showing that X contains all its limits points and hence is closed by utilizing the statements of Lemma 1 to Lemma 7. The existence of the result for the third order equation (1) was also established in Hilbert space by the extension principle and example was formulated which shows that a closed subset of a Hilbert space is a Hilbert Space.

CONCLUSION

In this study, a third order nonlinear FODEs was formulated and developed in a Hilbert space by the extension principle. The attractive property of these methods is that it gives a wider applicability of the results obtained which are found to be significant and contributing to knowledge since a third order linear and nonlinear fuzzy ordinary differential equation is defined and established in Hilbert space successfully which was not done in the existing literatures. It is recommended that future study should consider the development of system of third order linear and nonlinear fuzzy ordinary differential equations in Hilbert space by the extension principle.

CONTRIBUTION TO KNOWLEDGE

i. Third order nonlinear FODEs was formulated

ii. Development of new formulated ordinary differential equation in a Hilbert space by the extension principle.

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