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ORIGINAL RESEARCH ARTICLE

The properties of Type II Half-Logistic Exponentiated Weibull Distribution with Applications

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ABSTRACT
Recent research has demonstrated the utility of extending continuous distributions in fitting data of all kinds. This paper proposes the Type II Half-Logistic Exponentiated Weibull (TIIHLEtW) Distribution as a new distribution. For the Type II Half-Logistic Exponentiated Weibull distribution, we obtain precise expressions for the quantile function, probability-weighted, moments, moments generating function, reliability function, hazards function, and order statistics. The maximum likelihood estimation approach is used to estimate the parameters of the new distribution, and a simulation study is presented. Two real data sets are used to demonstrate the new distribution's applicability and flexibility. The findings indicated that the new distribution is a better fit for the data compared to the other models that were examined.

INTRODUCTION

A Breeds of distribution classes have been created by expanding or generalizing common continuous distributions. The generated family of continuous distributions is a new enhancement for developing and expanding classic distributions. The newly generated distributions have been broadly researched in various of fields, and providing greater application flexibility. The Weibull distribution is a widely used classic example of a lifetime distribution. It accurately exhibits a substantial variety of different failures, both in specific components and in general. Several generalizations and expansions of the Weibull distribution have been proposed in the statistical literature to address bathtub-shaped failure rates. Mudholkar et al., (1996) pioneered Exponentiated Weibull distribution, the modified Weibull extension by Xie et al., (2002) , flexible Weibull extension (FWEx) by Bebbington et al.,(2007), beta modified Weibull by Silva et al., (2010), Kumaraswamy Weibull by Cordeiro et al.,(2010), transmuted Weibull by Aryal and Tsokos (2011), truncated Weibull distribution by Zhang and Xie (2011), Kumaraswamy inverse Weibull by Shahbaz et al.,(2012), exponentiated generalized Weibull by Cordeiro et al.,(2013), McDonald modified Weibull by Merovci and Elbatal (2013), beta inverse Weibull by Hanook et al.,(2013), transmuted additive Weibull by Elbatal and Aryal (2013), McDonald Weibull by Cordeiro et al.,(2014), Kumaraswamy modified Weibull by Cordeiro et al.,(2014), transmuted complementary Weibull geometric by Afify et al.,(2014), Kumaraswamy transmuted exponentiated additive Weibull by Nofal et al.,(2016), generalized transmuted Weibull by Nofal et al.,(2017), Topp-Leone generated Weibull by Aryal et al.,(2017), Kumaraswamy complementary Weibull geometric by Afify et al.,(2017), Marshall-Olkin additive Weibull by Afify et al.,(2018), Zubair–Weibull by Ahmad (2018), alpha power transformed Weibull by Ahmad et al.,(2019), Topp Leone exponentiated weibull by Ibrahim (2021) distributions.

Bello et al., (2021) propound a new distribution family called the Type II Half-Logistic Exponentiated-G (TIHLEt-G) with two extra shape parameters. For any arbitrary cumulative distribution function as a baseline (cdf) , the TIIHLEt-G family with two positive shape parameters and has cumulative distribution function (cdf) and probability density function (pdf) given by:

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$$F_{TIIHLEt\_G}(x;\lambda,\alpha,\zeta)=\frac{2H^{\alpha\lambda}(x;\zeta)}{\left[1+H^{\alpha\lambda}(x;\zeta)\right]} \quad , \quad x>0,\ \lambda,\alpha>0 \tag{1}$$

and

$$f_{TIIHLEt-G}(x;\lambda,\alpha,\zeta)=\frac{2\lambda\alpha h(x;\zeta)H^{\alpha-1}(x;\zeta)\left[H^{\alpha(\lambda-1)}(x;\zeta)\right]}{\left[1+H^{\alpha\lambda}(x;\zeta)\right]^2},\ x>0,\ \lambda,\alpha>0 \tag{2}$$

The cdf and pdf of the Weibull distribution are given as

$$H(x;\theta,\beta)=1-e^{-\theta x^{\beta}},\ x>0,\theta,\beta>0 \tag{3}$$

$$h(x;\theta,\beta)=\theta\beta x^{\beta-1}e^{-\theta x^{\beta}},\ x>0,\theta,\beta>0 \tag{4}$$

According to Bello et al., (2021), the TIIHLEt-G distribution family has various characteristics that distinguish it from traditional distribution models. One such characteristic is its increased flexibility with regards to kurtosis, which can result in more accurately fitting the distribution to real data. Additionally, this family of distributions can produce skewed distributions even when the original data is symmetric. This can result in stronger margins and better-fitting models. The TIIHLEt-G distribution family is capable of generating different types of distributions, such as symmetric, left-skewed, right-skewed, and inverted J-shaped, allowing for the creation of models with varying hazard rate functions.

This paper intends to forge a more flexible model by extending the two-parameter Weibull distribution. The Type II half logistic Weibull (TIIHLEtW) distribution is the name given to the new model. We generate the TIIHLEtW distribution from Bello et al., (2021) and provide some essential statistical properties.

MATERIALS AND METHODS

The Type II Half-Logistic Exponentiated Weibull (TIIHLEtW) Distribution

We define a new model called the TIIHLEtW model, the random variable X is said to have a TIIHLEtW model, if its cdf is obtained by introducing equation (3) in equation (1) as follows:

$$F_{TIIHLEtW}(x;\lambda,\alpha,\theta,\beta)=\frac{2\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda}}{1+\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda}}\ ,x>0,\lambda,\alpha,\theta,\beta>0 \tag{5}$$

and its corresponding pdf is

$$f_{TIIHLEtW}(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha-1}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha(\lambda-1)}}{\left[1+\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda}\right]^2},x>0,\lambda,\alpha,\theta,\beta>0 \tag{6}$$

where β is a scale parameter and λ,α,θ are shape parameters.

Important Representation Important Representation

We presented a proper representation for the TIIHLEtW pdf and cdf. Due to the fact that the generalized binomial series is

$$\left(1+Z\right)^{-\tau}=\sum_{i=0}^{\infty}(-1)^i\binom{\tau+i-1}{i}z^i \tag{7}$$

For $|z|<1$ and τ is a positive real non integer. The density function of the TIIHLEtW distribution is then obtained using the binomial theorem (7) to (6).

$$f_{TIIHLEtW}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\sum_{i=0}^{\infty}(-1)^i\binom{1+i}{i}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda(i+1)-1}$$

Now, using the generalized binomial theorem, we can write

$$\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda(i+1)-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha\lambda(i+1)-1}{j}(e^{-\theta x^{\beta}})^{j}$$

Then, the pdf can be written as

$$f_{TIIHLEtW}(x;\lambda,\alpha,\theta,\beta)=\sum_{i,j=0}^{\infty}\eta_{p}(e^{-\theta x^{\beta}})^{j+1} \tag{8}$$

where, $\eta_{p}=2\alpha\lambda\theta\beta x^{\beta-1}(-1)^{i+j}\binom{1+i}{i}\binom{\alpha\lambda(i+1)-1}{j}$

Furthermore, an expansion for the $\left[F(x,\lambda,\alpha,\theta,\beta)\right]^{h}$ is produced, with h being an integer, and the binomial expansion is worked out once more.

$$\left[F(x;\lambda,\alpha,\theta,\beta)\right]^{h}=\sum_{k=0}^{h}(-1)^{k}\binom{h+k-1}{k}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda(h+k)}$$

Consider

$$\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda(h+k)}=\sum_{m=}^{\infty}(-1)^{m}\binom{\alpha\lambda(h+k)}{m}(e^{-\theta x^{\beta}})^{m}$$

The cdf can be written as:

$$\left[F(x;\lambda,\alpha,\theta,\beta)\right]^{h}=\sum_{k=0}^{h}\varphi_{t}(e^{-\theta x^{\beta}})^{m} \tag{9}$$

where

$$\varphi_{t}=2^{h}\sum_{m=0}^{\infty}(-1)^{k+m}\binom{h+k-1}{k}\binom{\alpha\lambda(h+k)}{m}$$

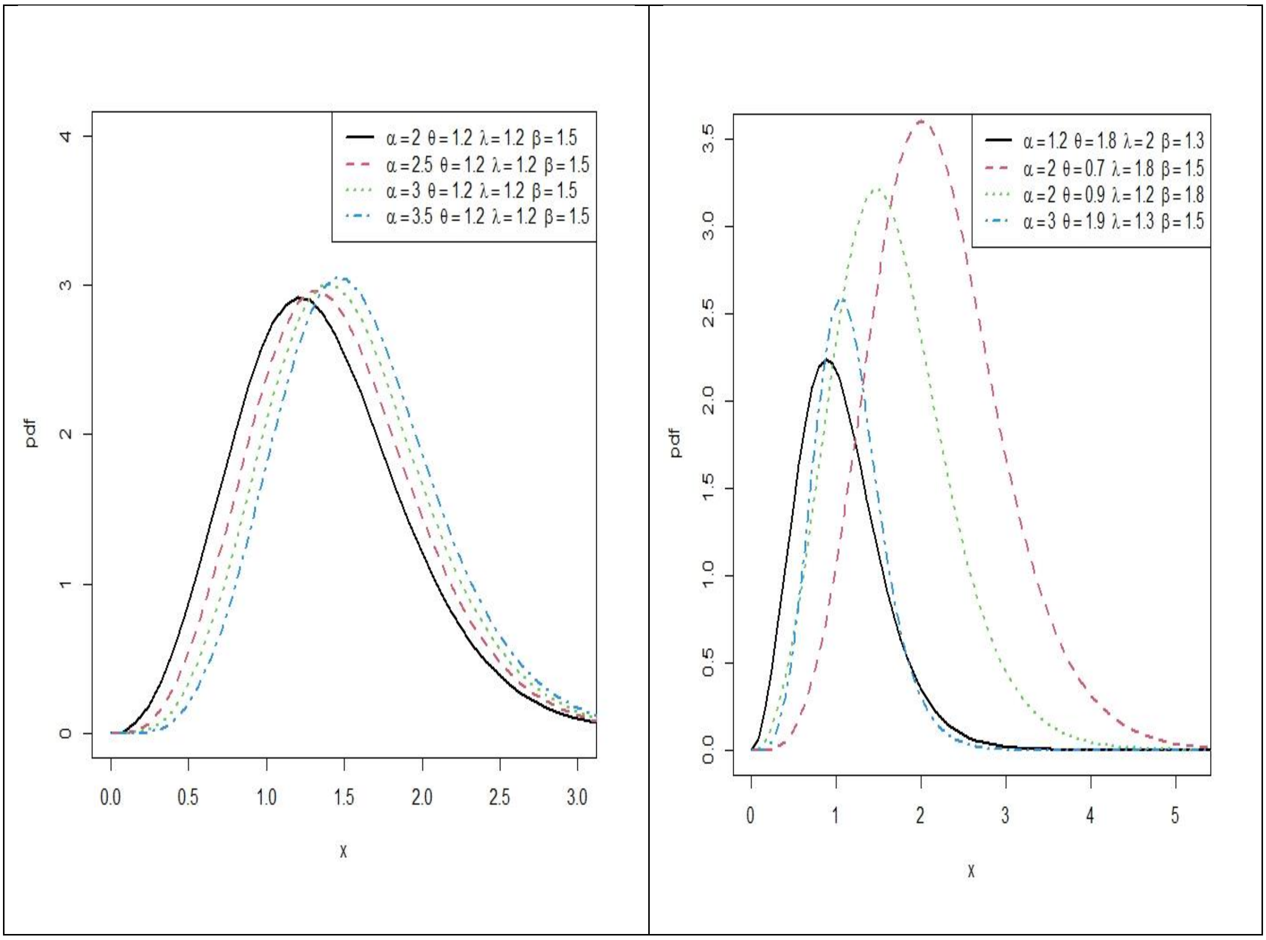

Figure 1: Plots of Pdf of TIIHLEtW distribution for different values of parameters

Statistical Properties

We derived some statistical properties of the new distribution.

Probability weighted moments

The probability-weighted moments (PWMs) were introduced by Greenwood *et al.,*(1979). It is used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, is denoted by $\kappa_{r,s}$ which can be derived for a random variable X using the following affiliations.

$$\kappa_{r,s} = E\left[X^r F(X)^s\right] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx \tag{10}$$

The PWMs of TIIHLEtW distribution is developed by substituting (8) and (9) into (10), and substituting h with s, as proceed

$$\kappa_{r,s} = \sum_{i,j=0}^{\infty}\sum_{k=0}^{s} \eta_p \varphi_t \int_0^{\infty} x^r (e^{-\theta x^{\beta}})^{j+1+m} dx \tag{11}$$

Consider the integral

$$\int_0^{\infty} x^r e^{-(j+1+m)\theta x^{\beta}} dx$$

Let $y=(j+1+m)\theta x^{\beta} \Rightarrow x=\left[\frac{y}{(j+1+m)\theta}\right]^{\frac{1}{\beta}}; dx=\frac{dy}{(j+1+m)\theta\beta x^{\beta-1}}$

Then

$$\int_0^{\infty}\left[\frac{y}{(j+1+m)\theta}\right]^{\frac{r}{\beta}} e^{-y}\frac{dy}{(j+1+m)\theta\beta x^{\beta-1}}=\frac{1}{\left[(j+1+m)\theta\right]^{\frac{r}{\beta}+1}}\int_0^{\infty} y^{\frac{r}{\beta}}e^{-y}dy$$

$$\int_0^{\infty} y^{\frac{r}{\beta}}e^{-y}dy=\Gamma(\frac{r}{\beta}+1)$$

The PWMs of TIIHLEtW can be written as proceed

$$\kappa_{r,s}=\sum_{i,j=0}^{\infty}\sum_{k=0}^{s}\frac{\eta_p\varphi_t\Gamma(\frac{r}{\beta}+1)}{(j+1+m)^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}} \tag{12}$$

now

$$\varphi_t=2^h\sum_{m=0}^{\infty}(-1)^{k+m}\binom{s+k-1}{k}\binom{\alpha\lambda(s+k)}{m}$$

and

$$\eta_p=2\lambda\alpha(-1)^{i+j}\binom{1+i}{i}\binom{\alpha\lambda(i+1)-1}{j}$$

Moments

Since the moments are essential in any statistical analysis, especially in applications. So, we derive the rth moment for the new distribution.

$$\mu_r^{'}=E(x^r)=\int_0^{\infty} x^r f(x)dx \tag{13}$$

By using the important representation of the pdf in equation (8), we have

$$E(X^r)=\sum_{i,j=0}^{\infty}\eta_p\int_0^{\infty} x^r(e^{-\theta x^{\beta}})^{j+1} \tag{14}$$

Consider the integral

$$\int_0^\infty x^r e^{-(j+1)\theta x^\beta} dx$$

Let $w=(j+1)\theta x^\beta \Rightarrow x=\left[\frac{w}{(j+1)\theta}\right]^{\frac{1}{\beta}}; dx=\frac{dw}{(j+1)\theta\beta x^{\beta-1}}$

Then

$$\int_0^\infty \left[\frac{w}{(j+1)\theta}\right]^{\frac{r}{\beta}} e^{-w}\frac{dw}{(j+1)\theta}=\frac{1}{\left[(j+1)\theta\right]^{r+1}}\int_0^\infty w^{\frac{r}{\beta}}e^{-w}dw=\int_0^\infty w^{\frac{r}{\beta}}e^{-w}dw=\Gamma(\frac{r}{\beta}+1)$$

The r[th] moment for TIIHLEtW distribution can be written as proceed

$$E(X^r)=\sum_{i,j=0}^{\infty}\frac{\eta_p\Gamma(\frac{r}{\beta}+1)}{(j+1)^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}} \tag{15}$$

Now

$$\eta_p=2\lambda\alpha(-1)^{i+j}\binom{1+i}{i}\binom{\alpha\lambda(i+1)-1}{j}$$

The mean and variance of TIHLEtW distribution are as follows

$$E(X)=\sum_{i,j=0}^{\infty}\frac{\eta_p\Gamma(\frac{1}{\beta}+1)}{(j+1)^{\frac{1}{\beta}+1}\theta^{\frac{1}{\beta}}} \tag{16}$$

and

$$\operatorname{var}\left(X\right)=\sum_{i,j=0}^{\infty}\frac{\eta_p\Gamma(\frac{1}{\beta}+1)}{(j+1)^{\frac{1}{\beta}+1}\theta^{\frac{1}{\beta}}}-\left[\sum_{i,j=0}^{\infty}\frac{\eta_p\Gamma(\frac{1}{\beta}+1)}{(j+1)^{\frac{1}{\beta}+1}\theta^{\frac{1}{\beta}}}\right]^2 \tag{17}$$

Moment-generating function (mgf)

The Moment-Generating Function of x is given as:

$$M_x\left(t\right)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f(x)dx \tag{18}$$

where the expansion of $e^{tx}=\sum_{m=0}^{\infty}\frac{t^m x^m}{m!}$

The moment-generating function of TIIHLEtW distribution is given by

$$M_x(t)=\sum_{i,j=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m\eta_p\Gamma(\frac{m}{\beta}+1)}{(j+1)^{\frac{m}{\beta}+1}\theta^{\frac{m}{\beta}}m!} \tag{19}$$

Reliability function

The reliability function gives the probability that a patient will survive longer than a specified time. It is defined as

$$R(x;\lambda,\alpha,\theta,\beta)=\frac{1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda}}{1+\left[1-e^{-\theta x^{\beta}}\right]^{\alpha\lambda}} \tag{20}$$

Hazard function

The hazard function is the probability of an event of interest occurring within a relatively short time frame and is defined as

$$T(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha-1}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha(\lambda-1)}}{1-\left[1-e^{-\theta x^{\beta}}\right]^{2\alpha\lambda}} \tag{21}$$

Quantile Function

A quantile function is a key tool for generating random variables from any continuous probability distribution. It, therefore, occupies an important place in probability theory. For x, the quantile function is F(x) = u, where u is distributed as U(0,1). The TIIHLEtW distribution is easily simulated by reversing equation (5), which leads the Quantile function Q(u), defined as:

$$x=Q(u)=\left\{\frac{-1}{\theta}\log\left[1-\left[\frac{U}{2-U}\right]^{\frac{1}{\alpha\lambda}}\right]\right\}^{\frac{1}{\beta}} \tag{22}$$

The first quartile, the median and the third quartile of TIIHLEtW distribution are obtained by putting u = 0.25, 0.5 and 0.75, respectively in equation (22).

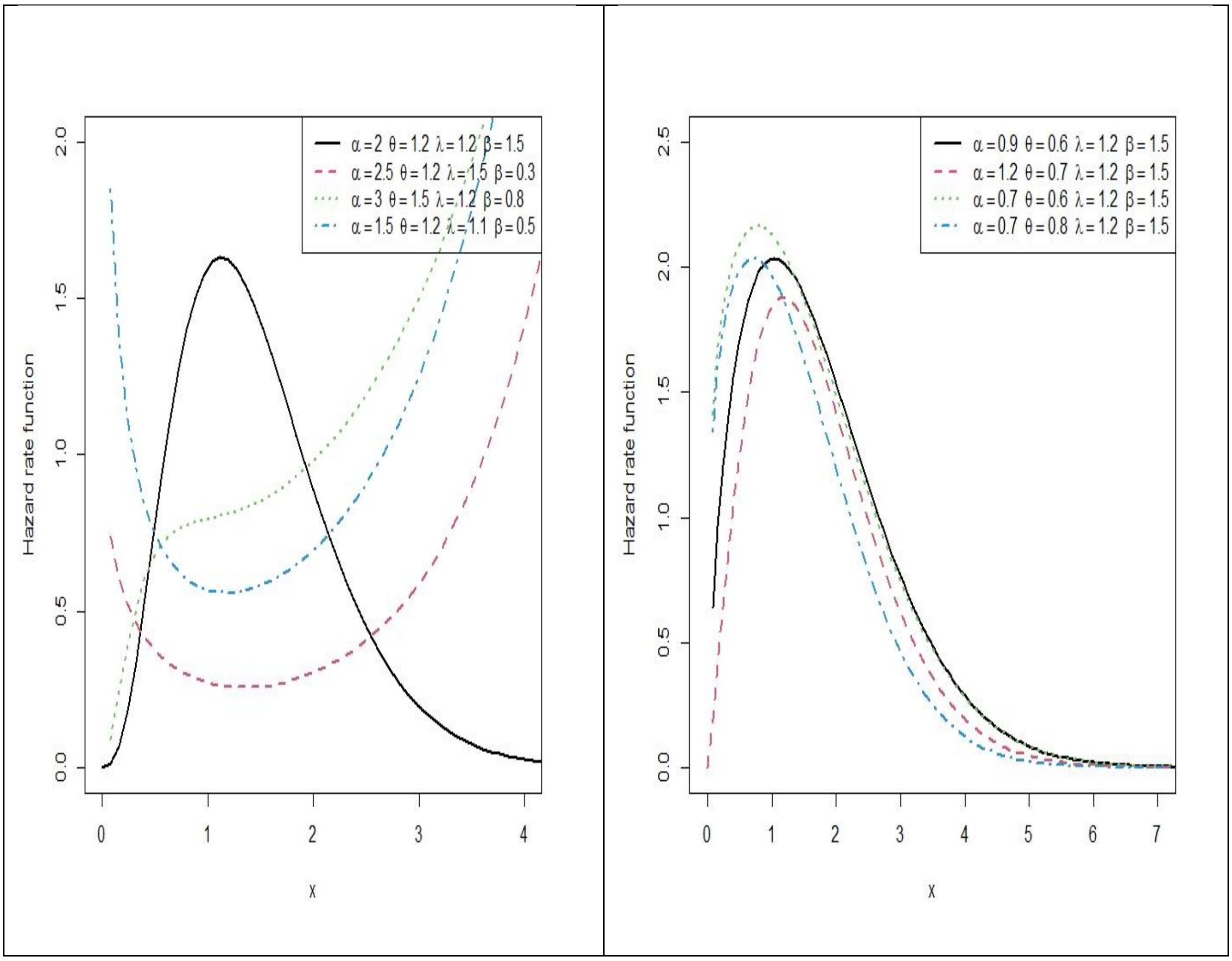

Figure 2: Plots of hazard of the TIIHLEtW distribution for different valves of parameters.

Order Statistics

Order statistics have been extensively used in various statistical fields, including reliability and life testing. Let X_1, X_2, ..., X_n be independent and identically distributed random variables with their corresponding continuous distribution function F (x). Let X_1, X_2,.., X_n be n independently distributed and continuous random variables from the TIIHLEtW distribution. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r = 1, 2, 3, ..., n denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. David (1970) gave the probability density function of $X_{r:n}$ as:

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{23}$$

By substituting equation (8) and equation (9) into equation (23), also replacing h with v+r-1 in equation (9). We have

$$f_{r:n}(x; \lambda, \alpha, \theta, \beta) = \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j=0}^{\infty} \sum_{k=0}^{r+v-1} (-1)^v \binom{n-r}{v} \eta_p \varphi_t (e^{-\theta x^\beta})^{j+1+m} \tag{24}$$

The equation above is called the r^{th} order statistics for the TIIHLEtW distribution.

Let r = n, then the probability density function of the maximum order statistics of TIIHLEtW distribution is

$$f_{n:n}(x;\lambda,\alpha,\theta,\beta)=2^{(n+v)}n\lambda\alpha\theta\sum_{i,j,m=0}^{\infty}\sum_{k=0}^{n+v-1}(-1)^{i+j+k+m+v}\binom{1+i}{i}\binom{\alpha\lambda(i+1)-1}{j}\binom{n+v+k-2}{k}\binom{\alpha\lambda(n+v-1+k)}{m}(e^{-\theta x^{\beta}})^{j+1+m} \tag{25}$$

Also, let r = 1, then the probability density function of the minimum order statistics of TIIHLEtW distribution is

$$f_{1:n}(x;\lambda,\alpha,\theta,\beta)=2^{(v+1)}n\lambda\alpha\theta\sum_{v=0}^{n-1}\sum_{i,j,m=0}^{\infty}\sum_{k=0}^{v}(-1)^{i+j+k+m+v}\binom{n-1}{v}\binom{1+i}{i}\binom{\alpha\lambda(i+1)-1}{j}\binom{v+k-1}{k}\binom{\alpha\lambda(v+k)}{m}(e^{-\theta x^{\beta}})^{j+1+m} \tag{26}$$

Parameter Estimation

Given complete data, we investigate the maximum likelihood method to estimate the TIIHLEtW distribution's unknown parameters. Maximum likelihood estimates (MLEs) are attractive because they can be used to produce confidence intervals and offer straightforward approximations that work well in finite samples. The resulting approximation for MLEs is simple to handle in distribution theory, both analytically and numerically. Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from the TIIHLEtW distribution. Then, the likelihood function based on the observed sample for the vector of the parameter $(\lambda, \alpha, \theta, \beta)^T$ is given by

$$\begin{aligned}\log L &= n\log(2)+n\log(\lambda)+n\log(\alpha)+n\log(\theta)+n\log\beta\\ &+(\beta-1)\sum_{i=1}^{n}\log(x_i)-\theta\sum_{i=1}^{n}(x_i^{\beta})+(\alpha-1)\sum_{i=1}^{n}\log[1-e^{-\theta x_i^{\beta}}]\\ &+\alpha(\lambda-1)\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i^{\beta}}\right]-2\sum_{i=1}^{n}\log\left[1+\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha\lambda}\right]\end{aligned} \tag{27}$$

The components of the score vector $\Delta L(\phi)=\left(\frac{\partial L(\phi)}{\partial\lambda},\frac{\partial L(\phi)}{\partial\alpha},\frac{\partial L(\phi)}{\partial\theta},\frac{\partial L(\phi)}{\partial\beta}\right)^{T}$ are given as

$$\frac{\partial\log L}{\partial\lambda}=\frac{n}{\lambda}+\alpha\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i^{\beta}}\right]-2\sum_{i=1}^{n}\frac{\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha\lambda}\log\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}}{1+\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha\lambda}} \tag{28}$$

$$\begin{aligned}\frac{\partial\log L}{\partial\alpha}&=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i^{\beta}}\right]+(\lambda-1)\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i^{\beta}}\right]\\ &-2\sum_{i=1}^{n}\frac{\lambda\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha(\lambda-1)}\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\log\left[1-e^{-\theta x_i^{\beta}}\right]}{1+\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha\lambda}}\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial\log L}{\partial\theta}&=\frac{n}{\theta}-\sum_{i=1}^{n}x_i^{\beta}+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\theta x_i^{\beta}}x_i^{\beta}}{1-e^{-\theta x_i^{\beta}}}+\alpha(\lambda-1)\sum_{i=1}^{n}\frac{e^{-\theta x_i^{\beta}}x_i^{\beta}}{1-e^{-\theta x_i^{\beta}}}\\ &-2\sum_{i=1}^{n}\frac{\lambda\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha(\lambda-1)}\alpha\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha-1}e^{-\theta x_i^{\beta}}x_i^{\beta}}{1+\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha\lambda}}\end{aligned} \tag{30}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) - \theta \sum_{i=1}^{n} x_i^{\beta} \log(x_i) + (\alpha - 1) \sum_{i=1}^{n} \frac{e^{-\theta x_i^{\beta}} \theta x_i^{\beta} \log(x_i)}{1 - e^{-\theta x_i^{\beta}}}$$

$$+\alpha(\lambda - 1) \sum_{i=1}^{n} \frac{e^{-\theta x_i^{\beta}} \theta x_i^{\beta} \log(x_i)}{1 - e^{-\theta x_i^{\beta}}} - 2 \sum_{i=1}^{n} \frac{\lambda \left[1 - e^{-\theta x_i^{\beta}}\right]^{\alpha(\lambda-1)} \alpha \left[1 - e^{-\theta x_i^{\beta}}\right]^{\alpha-1} e^{-\theta x_i^{\beta}} \theta x_i^{\beta} \log(x_i)}{1 + \left[1 - e^{-\theta x_i^{\beta}}\right]^{\alpha\lambda}} \quad (31)$$

The MLEs are obtained by setting $\frac{\partial L(\phi)}{\partial \lambda}, \frac{\partial L(\phi)}{\partial \alpha}, \frac{\partial L(\phi)}{\partial \theta}$ and $\frac{\partial L(\phi)}{\partial \beta}$ to zero and solving these equations simultaneously. These equations cannot be solved analytically, so we have to appeal to numerical method.

SIMULATION STUDY

In this section, a numerical analysis will be conducted to evaluate the performance of MLE for TIIHLEtW Distribution.

Table 1: MLEs, biases and RMSE for some values of parameters

		(1.5,1.5,3,3)			(2,0.5,1.5,1.5)		
n	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
	λ	2.3168	0.8168	0.9366	2.8261	0.8261	2.3271
20	α	1.9455	0.4455	1.3762	0.9257	0.4257	0.2824
	θ	3.1139	0.1139	0.6866	1.8902	0.3902	1.0447
	β	3.1139	0.1139	0.6866	1.8902	0.3902	1.0447
	λ	2.2450	0.7450	0.8161	2.5113	0.5113	1.9561
50	α	1.7841	0.2841	1.2438	0.7214	0.2214	0.2711
	θ	3.0153	0.0153	0.5803	1.7681	0.2681	0.8740
	β	3.0153	0.0153	0.5803	1.7681	0.2681	0.8740
100	λ	2.2035	0.7035	0.7587	2.1894	0.1894	1.5638
	α	1.6354	0.1354	0.7849	0.6129	0.1129	0.2627
	θ	3.0076	0.0076	0.4995	1.7075	0.2075	0.8012
	β	3.0076	0.0076	0.4995	1.7075	0.2075	0.8012
250	λ	2.1620	0.6620	0.7033	2.1749	0.1749	1.0828
	α	1.5811	0.0811	0.5484	0.5879	0.0879	0.2541
	θ	3.0051	0.0051	0.4153	1.5868	0.0868	0.6893
	β	3.0051	0.0051	0.4153	1.5868	0.0868	0.6893
500	λ	2.1195	0.6195	0.6607	2.0501	0.0501	0.7620
	α	1.5413	0.0413	0.3215	0.5692	0.0692	0.1492
	θ	3.0014	0.0014	0.4030	1.5413	0.0413	0.5355
	β	3.0014	0.0014	0.4030	1.5413	0.0413	0.5355
1000	λ	2.0808	0.5808	0.6161	2.0202	0.0202	0.4419
	α	1.5078	0.0078	0.1797	0.5479	0.0479	0.1173
	θ	3.0012	0.0012	0.3910	1.5343	0.0343	0.3910
	β	3.0012	0.0012	0.3910	1.5343	0.0343	0.3910

The table above shows that the values of biases and RMSEs approach zero. The estimates tend to the initial (true) values as the sample increases, indicating that the estimates are efficient and consistent.

RESULTS AND DISCUSSION

Applications to Real Data

We fit the TIIHLEtW distribution to two real data sets and give a comparative study with the fits to the Type II Half Logistic Weibull (TIIHLW) Distribution by Hassan *et al.,* (2019), Type II Exponentiated Half Logistic Weibull (TIIEHLW) distribution by Al-Mofleh *et al.,* (2020), Half-Logistic Generalized Weibull (HLGW) Distribution by Masood and Amna (2018), Exponentiated Weibull (EW) by Pal *et al.,* (2006), and Topp-Leone Generated Weibull (TLGW) Distribution by Aryal *et al.,* (2017) as comparator distributions for illustrative purposes.

The TIIHLW Distribution by Hassan *et al.,* (2019)

$$f(x;\theta,\lambda,\beta)=\frac{2\lambda\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\lambda-1}}{\left[1+\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{2}} \tag{32}$$

The TIIEHLW distribution developed by Al-Mofleh *et al.,* (2020) has pdf defined as:

$$f(x;\alpha,\lambda,\beta,\theta)=2\alpha\lambda\beta\theta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{\alpha-1}}{\left[1+\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{\alpha+1}} \tag{33}$$

The HLGW distribution developed by Masood and Amna (2018) has pdf defined as:

$$f(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x^{\alpha-1}\left[1+\theta x^{\alpha}\right]^{\lambda-1}exp\left[1-\left[1+\theta x^{\alpha}\right]^{\lambda}\right]}{\left[1+exp\left[1-\left[1+\theta x^{\alpha}\right]^{\lambda}\right]\right]^{2}} \tag{34}$$

The EW distribution proposed by Pal *et al.,* (2006) has pdf given as:

$$f(x;\alpha,\lambda,\beta)=\alpha\lambda^{\beta}\beta x^{\beta-1}\left[1-exp(-\lambda x)^{\beta}\right]^{\alpha-1}exp(-\lambda x)^{\beta} \tag{35}$$

The TLGW distribution developed by Aryal *et al.,* (2017) has pdf defined as:

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta\alpha-1}\left[1-\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta}\right]\left[2-\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta}\right]^{\alpha-1} \tag{36}$$

The two datasets used as illustrations in the application substantiate the new proposed distribution flexibility, applicability, and "best fit" when modeling the datasets empirically compared to the above comparator distributions. All of the calculations are executed using the R programming language.

Data set 1

The first data set shown below indicates the tensile strength of 100 observations of carbon fibers- previously used by Nicholas and Padgett (2006):

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

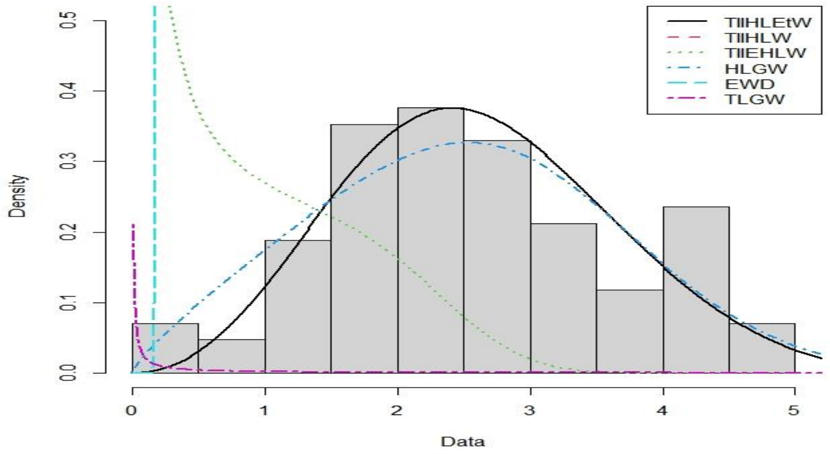

Figure 3: Fitted pdfs for the TIIHLEtW, TIIHLW, TIIEHLW, HLGW, EWD, and TLGW distributions to the data set 1.

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	β	**LL**	**AIC**
TIHLEtW	0.0106	1.0175	0.0118	3.5419	-142.734	293.4681
TIIHLW		1.2813	5.0457	0.5245	-145.6059	299.2117
TIIEHLW	0.1196	1.2483	0.7468	2.2646	-146.0227	300.0453
HLGW	1.4486	3.0294	0.0760		-146.6926	299.3853
EWD	7.5157	0.6959		1.4762	-144. 5191	295.0382
TLGW	3.7088	0.0087	9.5137	0.0862	-187.4776	382.9552

The parameters of the new proposed distribution and the five comparator distributions were estimated using maximum likelihood, and the results are shown in Table 2. The new proposed distribution reported the minimal AIC value according to the goodness of fit measure, though the TIIHLW was close behind. The proposed distribution's superiority over its rivals is further supported by a visual examination of the fit shown in Figure 3. The new recommended distribution thus "best fits" the carbon fibers data set among the distributions mentioned.

Data set 2

The second data set shown below represents the civil engineering data with 85 hailing times, previously used by Kotz and Dorp (2004):

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80.

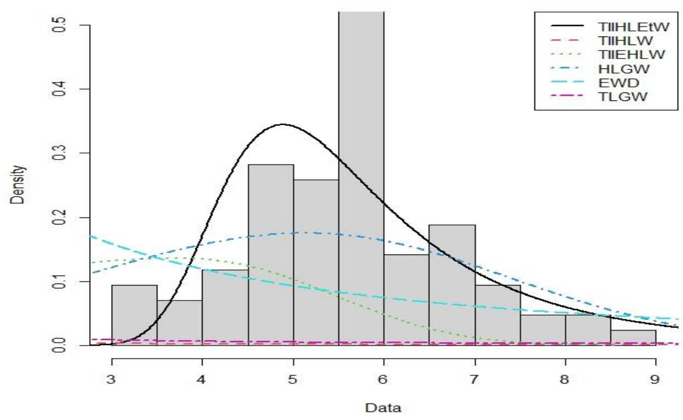

Figure 4: Fitted pdfs for the TIIHLEtW, TIIHLW, TIIEH LW, HLGW, EWD, and TLGW distributions to the data set 2.

Table 3: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	β	LL	AIC
TIIHLEtW	0.4587	3.2451	0.0004	4.4388	-164.8106	337.6211
TIIHLW		1.8379	1.6811	0.0358	-173.2528	354.5056
TIIEHLW	0.0508	0.2437	0.7486	1.7885	-196.6113	401.2226
HLGW	1.9017	1.1729	0.0432		-166.3807	338.7613
EWD	5.0124	2.3964		0.3562	-213.8578	433.7155
TLGW	27.3720	0.1182	0.0649	15.1151	-172.2809	340.5619

The parameters of the TIIHLEtW distribution and the five comparator distributions were estimated using maximum likelihood, as shown in Table 3. The new distribution reported the smallest AIC value based on the goodness of fit measure AIC, indicating that the distribution is the "best fit" for the hailing times. The new distribution's dominance over its competitors is confirmed by visually examining of the fit shown in Figure 4.

CONCLUSION

Using the family of distributions proposed by Bello *et al.,* (2021), we developed and investigated a novel distribution in this study known as the Type II Half-Logistic Exponentiated Weibull Distribution. As statistical elements of the new proposed distribution, explicit quantile function, probability weighted moments, moments, generating function, reliability function, hazard function, and order statistics were investigated. To estimate the parameters, the maximum likelihood method is used. We show some simulation findings to assess the proposed distribution's effectiveness. Two genuine data sets are assessed in comparison to well-known models to emphasize the significance and adaptability of the new distribution. According to the results, the new distribution seems superior to the other models that were properly considered, suggesting that it can be utilized to model data for various applications.

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