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ORIGINAL RESEARCH ARTICLE

Inventory Optimization for Non-Instantaneous Deteriorating Items Under Retailer's Trade Credit and Partial Backlogging

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ABSTRACT
In this research, an inventory model is developed for non-instantaneous deteriorating items with time-dependent demand under the retailer's trade credit and partial backlogging. In the model, the retailer receives no trade credit from the supplier but offers trade credit to customers. Payment strategies for the retailer were adopted in calculating the interest associated with the trade credit. The cost function was obtained and optimized using the gradient method. To illustrate the model, numerical examples are provided. The results show that case 1 scenario 2 with $t_1^* = 24.5070$ and $T^* = 31.7335$ produced the highest profit of $TP^* = 16{,}000$. Sensitivity analyses carried out to examine the influence of the inventory parameters showed that decreases in *h* by 10% and *N* by 20% lead to higher profit, indicating that decreases in holding costs and offering customers trade credit can lead to higher profitability and better decisions. From a managerial point of view, it is advised that retailers should regularly extend trade credit to customers to reduce item overstay and earn higher profits.

INTRODUCTION

Effective inventory management is the tracking of useful goods throughout the entire supply chain, from the purchase of raw materials to production to end sales. Thus, this achieves greater inventory visibility, meets customer demand, understands profitability, improves workflow efficiency, and enables accurate financial reporting. In addition, the nature of the inventory item to be managed plays a crucial role in the supply chain. Therefore, studying the inventory of deteriorating items is crucial owing to their dynamic nature.

Ghare and Schrader (1963) were the first to consider the optimal ordering policies for deteriorating items. They assumed the item to have a constant deterioration rate. Mandal and Phaujdar (1989) assumed a constant production rate and a linearly stock-dependent demand in an inventory model for deteriorating items. Dave and Patel (1981) presented an inventory model for deteriorating items with linearly increasing demand and a constant deterioration rate. Lee and Dye (2012) worked on an inventory model under stock-dependent demand and a controllable deterioration rate for deteriorating items. But instant deterioration does not always represent the nature of some deteriorating items. In practice, many perishable items such as fruits, vegetables, and meat do not deteriorate immediately upon receipt; instead, they exhibit a shelf-life period of no spoilage before deterioration sets in. This is termed in the literature as non-instantaneous deterioration.

Building on this, many stockists and researchers developed EOQ models for non-instantaneous deterioration, incorporating realistic features such as trade credit and shortages. For example, Wu et al. (2006) conducted a study to determine the optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand. Maihami and Kamalabadi (2012) considered joint pricing in an inventory model for non-instantaneous deteriorating items with permissible payment delays. Pal and Chandra (2014) developed an inventory model for non-instantaneous deteriorating items with stock- and time-dependent demand, while adopting a price discount to attract more sales. Pal and Samanta (2018) studied an inventory model with a non-instantaneous deterioration period, assuming the pre-deterioration period is random.

One of the management problems associated with keeping deteriorating items, whether instantaneous or non-instantaneous, is their nature. Keeping them for too long can lead to overstocking and further deterioration. Keeping less led to shortages, a phenomenon of vital importance. Therefore, it is worth noting that some works in the literature on non-instantaneous deteriorating items

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have studied shortages. When a shortages occur, it is more practical to treat unmet demand as partially backlogged rather than assuming full backlogging. Park (1982) studied an inventory model with partial backorder. Wee (1995) studied A deterministic lot-size inventory model for deteriorating items with shortages and a declining market. Mashud *et al.* (2017) studied an inventory model that considers price, stock-dependent demand with partially backlogged shortages, and two constant deterioration rates. Sundararajan *et al.* (2019) developed an inventory system of non-instantaneous deteriorating items with backlogging and time discounting.

In the earliest models, inventory was assumed to be paid for upon receipt of consignments (Goyal, 1985). However, in today's competitive market, businesses use trade credit to stimulate demand and increase profitability. Trade credit has typically been analyzed at the supplier level or at the two-level trade credit level. Goyal (1985) was the first to propose an EOQ model that incorporates a permissible payment delay. The supplier extended trade credit to the retailer. Pal and Maity (2012) developed an inventory model for deteriorating items with a constant deterioration rate and a trade credit policy. Shortage is allowed and completely backlogged. Chandra et al. (2020) considered optimal replenishment strategies for a constant-demand item under a progressive trade credit policy, accounting for non-instantaneous deterioration. Babangida and Baraya (2022) developed a model for non-instantaneous deteriorating items with two-phase demand rates, time-dependent linear holding costs, and shortages under a a trade credit policy. Vadana and Sharma (2016), proposed an inventory model for non-instantaneous deterioration items under a quadratic demand rate with permissible delay in payment and time dependent deterioration rate. In the model, unmet demand is fully backlogged. Vadana and Sharma (2016) modeled non-instantaneous deterioration with quadratic demand and permissible delay in payment, but assumed full backlogging.

In other studies, such as Aliyu (2022), trade credit is extended from the supplier–retailer level to the retailer–customer level. This is termed as two-level trade credit. Despite the prevalence of this practice of downstream trade credit in the market, it was only recently that Malumfashi et al. (2023) justified that it is worthwhile to consider the trade credit to be between the retailer and the customers only by proposing an inventory model for deteriorating items having linear time-dependent demand under downstream trade credit. In the model, the supplier offers no trade credit to the retailer for goods ordered, and the retailer offers trade credit to the customer for a period. However, this idea of downstream has not been studied when the items are non-instantaneously deteriorating. This needs to be addressed and that's the theme of this work.

NOTATIONS AND ASSUMPTIONS

Notations

The following are the notations used in the model:

$D(t)$ – is the time-varying demand of items. It is given as $D(t) = \mathrm{a} + \mathrm{bt}$ where $a > 0$ is the initial demand and $b > 0$ is the demand rate.

I_o – is the maximum inventory ordered by the retailer.

$P,\ C$ – are the selling price per unit and purchasing cost per unit item, respectively.

$I_p,\ I_e$ – are the interest payable and interest earned return rates paid and received by the retailer, respectively.

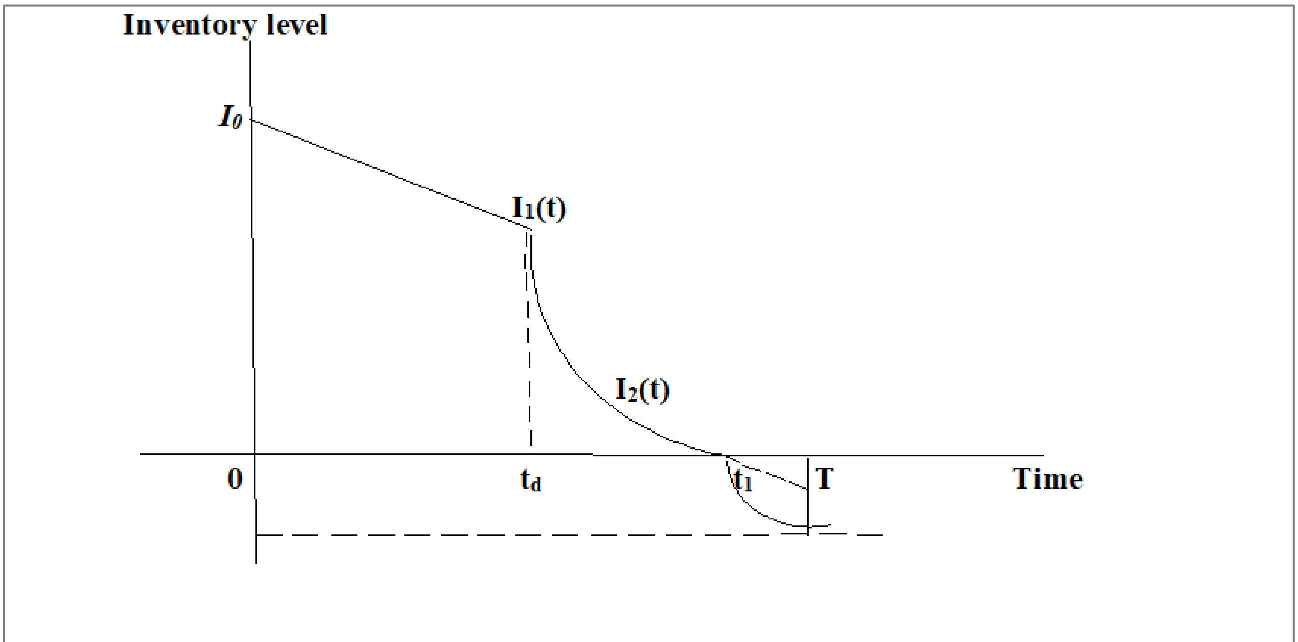

Figure 1: Pictorial representation of the model.

$\theta(t) -$ is the deterioration rate of the item.

h $-$ is the holding cost per unit item per unit time.

σ, S $-$ are the backlogging rate and quantity of maximum backlogged demands.

t_d, t_1 $-$ are the times when deterioration sets in and the time in which inventory finishes, respectively.

T $-$ is the end of the replenishment cycle.

A $-$ is the ordering cost per order.

$I_1(t), I_2(t)$, $I_3(t)$ – are the inventory levels at a time t, $0 \leq t \leq T$, before deterioration sets in, after deterioration sets in, and during the shortage period.

N $-$ The retailer's trade credit period offered to the customer.

TP $-$ The optimal total profit per unit time of the inventory system.

Assumptions

The following assumptions are made in the model:

i. The retailer offers trade credit to the customers.
ii. The supplier offers no trade credit to the retailer.
iii. Shortages are allowed and partially backlogged.
iv. The item is single and non-instantaneously deteriorating.
v. The replenishment rate is infinite and instantaneous.
vi. The lead time is zero.
vii. The demand is time-dependent.
viii. We restrict N > t_d

MATHEMATICAL MODEL FORMULATION

At the beginning of the cycle, I_0 Units of items are stocked in the inventory. The items maintained freshness during the period $(0, t_d)$ After which, the items start to deteriorate. The deterioration and demand continued during the period (t_d, t_1) till when the stocked items are depleted completely. At this time, shortages are occurring because demand is ongoing, indicating the business cycle has not ended. Therefore, from t_1 to *T*, the unmet demands are partially backlogged. Figure 1 represents the scenario. This phenomenon is modelled as follows:

$$\frac{dI_1(t)}{dt} = -(a + bt) \qquad 0 < t \leq t_d \tag{1}$$

with initial condition $I_1(0) = I_0$

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -(a + bt) \qquad t_d < t \leq t_1 \tag{2}$$

with condition $I_2(t_1) = 0$

$$\frac{dI_3(t)}{dt} = -\sigma(a + bt) \qquad t_1 < t < T \tag{3}$$

with condition $I_3(t_1) = 0$

The solutions to equation (1), (2) and (3) using the initial and boundary condition are given as

$$I_1(t) = I_0 - at - \frac{1}{2}bt^2 \qquad 0 < t \leq t_d \tag{4}$$

$$I_2(t) = \frac{1}{\theta^2}\left[(\mathrm{b} - \theta(a + bt)) - (\mathrm{b} - \theta(a + bt_1))e^{\theta(t_1 - t)}\right] \qquad t_d < t \leq t_1 \tag{5}$$

$$I_3(t) = \sigma a(t_1 - t) + \sigma\frac{b}{2}({t_1}^2 - t^2) \qquad t_1 < t < T \tag{6}$$

For the sales to continue after deterioration sets in, a continuity is established at $t = t_d$, by equating equations (4) and (5) and is given as:

$$I_o = \frac{1}{\theta^2}\left[\left(b - \theta(a + bt_d)\right) - \left(b - \theta(a + bt_1)\right)e^{\theta(t_1 - t_d)}\right] + at_d + \frac{1}{2}b{t_d}^2 \tag{7}$$

To get the total profit (TP) per annum, we take the difference between the sales revenue per annum with the total incurred costs during the annum. Therefore, we first calculate the following costs

i. Annual ordering cost (OC)

The ordering cost per order is given as *A*, therefore, OC per annum is given as

$$OC = \frac{A}{T} \tag{8}$$

ii. Annual holding cost (HC)

The HC is giving as $h\int_0^T I(t)dt = h\int_0^{t_1} I_1(t)dt + h\int_{t_1}^T I_2(t)dt$

After evaluating the definite integral, the annual HC is giving by

$$HC = \frac{h}{T}\Big[\Big\{\frac{1}{\theta^2}\big(b-\ \theta(a+bt_d)\big) - \frac{1}{\theta^2}\big(\mathrm{b}-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)}\Big\}t_d + at_d^2 + \frac{1}{2}\,bt_d^3 - \frac{at_d^2}{2} - \frac{1}{6}bt_d^3 + \frac{1}{\theta^2}\Big(b(\,t_1-\ t_d) - \theta a(t_1-t_d)\ - \frac{1}{2}\theta b(t_1^2-t_d^2)\Big) - \Big(b-\theta(a+bt_1)\frac{1}{\theta}\big(e^{\theta(t_1-t_d)} - 1\big)\Big)\Big] \quad (9)$$

iii. Annual Deteriorating cost (DC)

The number of deteriorated items is $\theta\int_{t_1}^T I_2(t)dt$. The annual DC incurred by the retailer, after some algebraic simplification, is giving as

$$DC = \frac{C\theta}{T}\Big[\frac{1}{\theta^2}\Big(b(\,t_1-\ t_d) - \theta a(t_1-t_d)\ - \frac{1}{2}\,\theta b(t_1^2-t_d^2)\Big) - \Big(b-\theta(a+bt_1)\frac{1}{\theta}\big(e^{\theta(t_1-t_d)} - 1\big)\Big)\Big] \quad (10)$$

iv. Annual Shortage Cost (SC)

The cost due to shortages incurred by the retailer is given as

$$SC = \int_{t_1}^T - I_3(t)\,dt =$$

$$c\left[\sigma a\left(\frac{T^2}{2} - Tt_1 + \frac{t_1^2}{2}\right) + \frac{\sigma b}{6}(T^3 - 3T{t^2}_1 + 2t_1^2)\right] \quad (11)$$

v. Annual Opportunity Cost (OP)

The annual OP incurred by the retailer due to lost sales is giving us

$$OP = \int_{t_1}^T (1-\sigma)(a+bt)dt$$

$$= (1-\sigma\,)\left(aT + \frac{bT^2}{2} - at_1 - \frac{bt_1^2}{2}\right) \quad (12)$$

vi. Purchasing cost (PC)

The cost of purchasing the stocked items is given as

$$PC = C\Big[-\Big(\sigma a(t_1-T) + \frac{\sigma b}{2}\,(t_1^2-\ T^2)\Big) + \frac{1}{\theta^2}\big(\mathrm{b}-\theta(a+bt_d)\big) - \frac{1}{\theta^2}\big(\mathrm{b}-\theta(a+bt_1)e^{\theta(t_1-t_d)}\big) + at_d + \frac{1}{2}b{t_d}^2\Big] \quad (13)$$

vii. Annual Sales Revenue (SR)

The sales revenue generated by the retailer due to sales during the period $(0,\ T)$ is giving as

$$SR = \frac{P}{T}\left[\int_0^{t_1}(a+bt)dt - \Big(\sigma a(t_1-T) + \frac{\sigma b}{2}\,(t_1^2-\ T^2)\Big)\right] = \frac{P}{T}\left[at_1 + \frac{bt_1^2}{2} - \Big(\sigma a(t_1-T) + \frac{\sigma b}{2}\,(t_1^2-\ T^2)\Big)\right] \quad (14)$$

viii. Interest Earned (I_E) and Interest Payable (I_P) by the retailer

Since the retailer did not get the grace of trade credit but gave the grace to his/her customers for a period N, then two cases are possible:

Case 1: when $N < T$ (when the trade credit period given to customers is less than the replenishment cycle) and

Case 2: when $N \geq T$ (when the trade credit period offered to the customers exceeds the replenishment cycle).

Now,

Case 1: N < T

In this case, the supplier offers no trade credit to the retailer, then the annual total interest payable by the retailer is given as:

$$I_{P1} = 0 \tag{15}$$

For the I_E, since there are still unsold items with the customer at the expiration of the trade credit period given, then the retailer may earn interest in two ways:

Scenario 1: when partial payment is acceptable by the retailer from the customer, and it is to be made at $t = N$ and the rest amount at a time after $t = N$.

Here, we assume the customer pays some amount (partial payment) at $t = N$ to the retailer and will pay the remaining amount C – C$_R$ after N ($t > N$). The total amount paid by customer C$_R$ during the period [0, N] is

$$C_R = C\int_0^N D(t)dt = C\left[aN + \frac{bN^2}{2}\right] \tag{16}$$

Let the rest amount to be paid to the retailer at time $t = M$ be $(CI_0 - C_R)$, where $M > N$. Then total annual interest payable by the customer to the retailer at time t = M is given as

$$I_{E11} = \frac{1}{T}[(CI_0 - C_R) + I_e(CI_0 - C_R)(M - N)] = \frac{1}{T}[(CI_0 - C_R)(1 + I_e(M - N))]$$

After some algebraic simplifications,

$$I_{E11} = \frac{C}{T}\left[\left(\frac{1}{\theta^2}\left(\mathrm{b} - \theta(a + bt_d)\right) - \frac{1}{\theta^2}\left[\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\right] + at_d + \frac{1}{2}bt_d{}^2 - \left[aN + \frac{bN^2}{2}\right]\right)(1 + I_e(M - N))\right] \tag{17}$$

Therefore, the *TP* for case 1 scenario 1 is

$$TP_{11} = \frac{1}{T}(SR + I_{E11} - OC - HC - DC - I_{P1} - SC - OP - PC)$$

Using equations (8), (9), (10), (11), 12), (13), 14), (15), and (17) and simplifying

$$TP_{11} = \frac{1}{T}\Big\{P\left[at_1 + \frac{bt_1^2}{2} - \left(\sigma a(t_1 - T) + \frac{\sigma b}{2}(t_1^2 - T^2)\right)\right] + C\left[\left(\frac{1}{\theta^2}\left(\mathrm{b} - \theta(a + bt_d)\right) - \frac{1}{\theta^2}\left[\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\right] + at_d + \frac{1}{2}bt_d{}^2 - \left[aN + \frac{bN^2}{2}\right]\right)(1 + I_e(M - N))\right] - A - h\Big[\Big\{\frac{1}{\theta^2}\left(b - \theta(a + bt_d)\right) - \frac{1}{\theta^2}\left(\mathrm{b} - \theta(a + bt_1)\right)e^{\theta(t_1 - t_d)}\Big\}t_d + at_d^2 + \frac{1}{2}bt_d^3 - \frac{at_d^2}{2} - \frac{1}{6}bt_d^3 + \frac{1}{\theta^2}\left(b(t_1 - t_d) - \theta a(t_1 - t_d) - \frac{1}{2}\theta b(t_1^2 - t_d^2)\right) - \left(b - \theta(a + bt_1)\frac{1}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right)\Big] - c\theta\left[\frac{1}{\theta^2}\left(b(t_1 - t_d) - \theta a(t_1 - t_d) - \frac{1}{2}\theta b(t_1^2 - t_d^2)\right) - \left(b - \theta(a + bt_1)\frac{1}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right)\right] - c\left[\sigma a\left(\frac{T^2}{2} - Tt_1 + \frac{t_1^2}{2}\right) + \frac{\sigma b}{6}(T^3 - 3Tt^2{}_1 + 2t_1^2)\right] - c\left[(1 - \sigma)\left(aT + \frac{bT^2}{2} - at_1 - \frac{bt_1^2}{2}\right)\right] - C\left[-\left(\sigma a(t_1 - T) + \frac{\sigma b}{2}(t_1^2 - T^2)\right) + \frac{1}{\theta^2}\left(\mathrm{b} - \theta(a + bt_d)\right) - \frac{1}{\theta^2}\left(\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\right) + at_d + \frac{1}{2}bt_d{}^2\right]\Big\} \tag{18}$$

Scenario 2: (when partial payment is not accepted at t >N)

In this scenario, due to the retailer not willing to accept partial payment from the consumer, full payment is to be made after $t = N$. Let full payment be made at time $t = M_2$, where $M_2 > N$. Therefore, the interest payable I_{E12} to the retailer by the consumer is given by

$$I_{E12} = CI_0 + CI_0I_e(M - N) = CI_0\left(1 + I_e(M - N)\right)$$

Substituting I_0 in equation (7), we see that

$$I_{E12} = C\left[\left(\frac{1}{\theta^2}\left(\mathrm{b} - \theta(a + bt_d)\right) - \frac{1}{\theta^2}\left[\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\right] + at_d + \frac{1}{2}bt_d{}^2\right)(1 + I_e(M - N))\right] \tag{19}$$

Therefore, the *TP* for case 1 scenario 2 is given by

$$TP_{12} = \frac{1}{T}(SR + I_{E12} - OC - HC - DC - I_{P1} - SC - OP - PC)$$

Using equation (8), (9), (10), (11), (12), (13), (14), (15) and (19) we get

$$TP_{12} = \frac{1}{T}\Big\{P\Big[at_1 + \frac{bt_1^2}{2} - \Big(\sigma a(t_1 - T) + \frac{\sigma b}{2}(t_1^2 - T^2)\Big)\Big] + C\Big[\Big(\frac{1}{\theta^2}(\mathrm{b} - \theta(a + bt_d)) - \frac{1}{\theta^2}\big[\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\big] + at_d + \frac{1}{2}bt_d{}^2\Big)\ (1 + I_e(M - N))\Big] - A - h\Big[\Big(\frac{1}{\theta^2}(b - \theta(a + bt_d)) - \frac{1}{\theta^2}(\mathrm{b} - \theta(a + bt_1))e^{\theta(t_1 - t_d)}\Big)t_d + at_d^2 + \frac{1}{2}bt_d^3 - \frac{at_d^2}{2} - \frac{1}{6}bt_d^3 + \frac{1}{\theta^2}\Big(b(t_1 - t_d) - \theta a(t_1 - t_d) - \frac{1}{2}\theta b(t_1^2 - t_d^2)\Big) - \Big(b - \theta(a + bt_1)\frac{1}{\theta}\big(e^{\theta(t_1 - t_d)} - 1\big)\Big)\Big] - c\theta\Big[\frac{1}{\theta^2}\Big(b(t_1 - t_d) - \theta a(t_1 - t_d) - \frac{1}{2}\theta b(t_1^2 - t_d^2)\Big) - \Big(b - \theta(a + bt_1)\frac{1}{\theta}\big(e^{\theta(t_1 - t_d)} - 1\big)\Big)\Big] - c\Big[\sigma a\Big(\frac{T^2}{2} - Tt_1 + \frac{t_1^2}{2}\Big) + \frac{\sigma b}{6}(T^3 - 3Tt^2{}_1 + 2t_1^2)\Big] - c\Big[(1 - \sigma)\Big(aT + \frac{bT^2}{2} - at_1 - \frac{bt_1^2}{2}\Big)\Big] - C\Big[-\Big(\sigma a(t_1 - T) + \frac{\sigma b}{2}(t_1^2 - T^2)\Big) + \frac{1}{\theta^2}(\mathrm{b} - \theta(a + bt_d)) - \frac{1}{\theta^2}\big(\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\big) + at_d + \frac{1}{2}bt_d{}^2\Big]\Big\} \tag{20}$$

Case 2: when $N \geq T$

Since the supplier offers no trade credit to the retailer, then the annual total interest payable by the retailer for this case says I_{P2} is given by

$$I_{P2} = 0 \tag{21}$$

On the other hand, since $N \geq T$, the customer pays no interest to the retailer. Therefore, the interest earned by the retailer says I_{e2} is given by

$$I_{E2} = 0 \tag{22}$$

Therefore, to get the total profit relevant per cycle for case 2

$$TP_2 = \frac{1}{T}(SR + I_{E2} - OC - HC - DC - I_{P2} - SC - OP - PC)$$

Using equation (8), (9), (10), (11), (12), (13), (14), (21) and (22) we get

$$TP_2 = \frac{1}{T}\Big\{P\Big[at_1 + \frac{bt_1^2}{2} - \Big(\sigma a(t_1 - T) + \frac{\sigma b}{2}(t_1^2 - T^2)\Big)\Big] - A - h\Big[\Big(\frac{1}{\theta^2}(b - \theta(a + bt_d)) - \frac{1}{\theta^2}(\mathrm{b} - \theta(a + bt_1))e^{\theta(t_1 - t_d)}\Big)t_d + at_d^2 + \frac{1}{2}bt_d^3 - \frac{at_d^2}{2} - \frac{1}{6}bt_d^3 + \frac{1}{\theta^2}\Big(b(t_1 - t_d) - \theta a(t_1 - t_d) - \frac{1}{2}\theta b(t_1^2 - t_d^2)\Big) - \Big(b - \theta(a + bt_1)\frac{1}{\theta}\big(e^{\theta(t_1 - t_d)} - 1\big)\Big)\Big] - c\theta\Big[\frac{1}{\theta^2}\Big(b(t_1 - t_d) - \theta a(t_1 - t_d) - \frac{1}{2}\theta b(t_1^2 - t_d^2)\Big) - \Big(b - \theta(a + bt_1)\frac{1}{\theta}\big(e^{\theta(t_1 - t_d)} - 1\big)\Big)\Big] - c\Big[\sigma a\Big(\frac{T^2}{2} - Tt_1 + \frac{t_1^2}{2}\Big) + \frac{\sigma b}{6}(T^3 - 3Tt^2{}_1 + 2t_1^2)\Big] - c\Big[(1 - \sigma)\Big(aT + \frac{bT^2}{2} - at_1 - \frac{bt_1^2}{2}\Big)\Big] - C\Big[-\Big(\sigma a(t_1 - T) + \frac{\sigma b}{2}(t_1^2 - T^2)\Big) + \frac{1}{\theta^2}(\mathrm{b} - \theta(a + bt_d)) - \frac{1}{\theta^2}\big(\mathrm{b} - \theta(a + bt_1)e^{\theta(t_1 - t_d)}\big) + at_d + \frac{1}{2}bt_d{}^2\Big]\Big\} \tag{23}$$

OPTIMIZATION ANALYSIS

Considering the two decision variables t_1 and T,

The necessary conditions for TP_{11} to be maximized are $\frac{\partial TP_{11}}{\partial t_1} = 0\ and\ \frac{\partial TP_{11}}{\partial T} = 0.$

Using equation (18)

$$\frac{\partial TP_{11}}{\partial T} = \frac{1}{T}\Big\{P[\sigma a + \sigma bT] - C\Big[\sigma a(T - t_1) + \frac{\sigma b}{6}(3T^2 - 3t_1^2)\Big] - c[(1 - \sigma)(a + bT)] - c[\sigma a + \sigma bT] - TP_{11}\Big\} = 0 \tag{24}$$

and

$$\frac{\partial TP_{11}}{\partial t_1} = \frac{1}{T}\Big\{P[(a+bt_1)-(\sigma a+\sigma bt_1)] + c\Big[-\frac{1}{\theta^2}\Big((-\theta b)e^{\theta(t_1-t_d)} + \theta\big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)}\Big)\big(1+I_e(M-N)\big)\Big] - h\Big[\frac{-1}{\theta^2}\big(\theta\big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)} - \theta b e^{\theta(t_1-t_d)}\big)t_d + \frac{1}{\theta^2}(b-\theta a-\theta bt_1) - \big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)} + b\big(e^{\theta(t_1-t_d)}-1\big)\Big] - c\theta\Big[\frac{1}{\theta^2}(b-\theta a-\theta bt_1) - (b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + b\big(e^{\theta(t_1-t_d)}-1\big)\Big] - c[(1-\sigma)(-a-bt_1)] - c\Big[\sigma a(-T+t_1) + \frac{\sigma b}{6}(-6Tt_1+4t_1)\Big] - c\Big[-(\sigma a+\sigma bt_1) - \frac{1}{\theta^2}\Big((-\theta b)e^{\theta(t_1-t_d)} + \theta\big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)}\Big)\Big]\Big\} = 0 \quad (25)$$

The solution to the non-linear equation (24) and (25) gives the value of t_1 and T. To confirm that the optimal solution $(t_1^*T^*)$ exist and unique, we show that

$\left(\frac{\partial^2 TP_{11}}{\partial t_1^2}\right) < 0$, $\left(\frac{\partial^2 TP_{11}}{\partial T^2}\right) < 0$ and $\left(\frac{\partial^2 TP_{11}}{\partial T^2}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_{11}}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1\partial T}\right)|_{(t_1^*T^*)} > 0$

Differentiating equation (24) further

$$\frac{\partial^2 TP_{11}}{\partial T^2} = \frac{1}{T}\Big\{P[\sigma b] - c[\sigma a+\sigma bT] - c[b(1-\sigma)] - c[\sigma b] - 2\frac{\partial TP_{11}}{\partial T}\Big\} \quad (26)$$

Evaluating equation (26) at (t_1^*, T^*), and since $\frac{\partial TP_{11}}{\partial T}\}|_{(t_1^*T^*)} = 0$, we see that

$$\frac{\partial^2 TP_{11}}{\partial T^2}|_{(t_1^*T^*)} < 0 \quad (27)$$

Now differentiating equation (25) with respect to t_1, we see that,

$$\frac{\partial^2 TP_{11}}{\partial t_1^2} = \frac{1}{T}\Big\{P[b-\sigma b] + c\Big[\frac{-1}{\theta^2}\big(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)}\big)\big(1+I_e(M-N)\big)\Big] - \mathrm{h}\Big[\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1)e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})t_d + \frac{1}{\theta}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)}\Big] - c\theta\Big[\frac{1}{\theta^2}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)}\Big] - c[(-b)(1-\sigma)] - c\Big[\sigma a + \frac{\sigma b}{6}(4-6T)\Big] - c\Big[-(\sigma b)\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})\Big]\Big\} \quad (28)$$

When equation (28) is evaluated at $(t_1^*T^*)$

$$\frac{\partial^2 TP_{11}(t_1^*T^*)}{\partial t_1^2} < 0 \quad (29)$$

Likewise,

$$\frac{\partial^2 TP_{11}}{\partial T\partial t_1} = \frac{1}{T}\{c\sigma[a+bt_1]\} \quad (30)$$

Also, from equation (25)

$$\frac{\partial^2 TP_{11}}{\partial t_1\partial T} = \frac{1}{T}\{c\sigma[a+bt_1]\} \quad (31)$$

Also,

$$\left(\frac{\partial^2 TP_{11}}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1\partial T}\right) = \frac{1}{T}\{c\sigma[a+bt_1]\}^2 > 0 \quad (32)$$

Evaluating,

$$\left(\frac{\partial^2 TP_{11}}{\partial T^2}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_{11}}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1\partial T}\right) = \Big(-\frac{1}{\mathrm{T}}\{P[\sigma b] - c[\sigma a+\sigma bT] - c[b(1-\sigma)] - c[\sigma b]\}\Big)\Big(\frac{1}{T}\Big\{P[b-\sigma b] + c\Big[\frac{-1}{\theta^2}\big(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)}\big)\big(1+I_e(M-N)\big)\Big] - \mathrm{h}\Big[\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1)e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})t_d + \frac{1}{\theta}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)}\Big] - c\theta\Big[\frac{1}{\theta^2}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} +$$

$$\theta be^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)}\Big] - c[(-b)(1-\sigma)] - c\left[\sigma a + \frac{\sigma b}{6}(4-6T)\right] - c\left[-(\sigma b)\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})\right]\Big\}\Big) - \left(\frac{1}{T}\{c\sigma[a+bt_1]\}^2\right) > 0 \quad (33)$$

Since $\left(\frac{\partial^2 TP_{11}}{\partial t_1^2}\right) < 0, \left(\frac{\partial^2 TP_{11}}{\partial T^2}\right) < 0$ and $\left(\frac{\partial^2 TP_{11}}{\partial T^2}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_{11}}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TP_{11}}{\partial t_1 \partial T}\right) > 0$ We conclude that TP_{11}^* at the point t_1^* and T^* is a maximum. Therefore equation (27) (29) (30) and (31) satisfied the Hessian matrix. Therefore TP_{11} is negative definite.

The necessary conditions for TP_{12} to be maximized are $\frac{\partial TP_{12}}{\partial t_1} = 0$, $\frac{\partial TP_{12}}{\partial T} = 0.$

Using equation (28)

$$\frac{\partial TP_{12}}{\partial T} = \frac{1}{T}\left\{P[\sigma a + \sigma bT] - C\left[\sigma a(T-t_1) + \frac{\sigma b}{6}(3T^2 - 3t_1^2)\right] - c[(1-\sigma)(a+bT)] - c[\sigma a + \sigma bT] - TP_{12}\right\} = 0 \quad (34)$$

And

$$\frac{\partial TP_{12}}{\partial t_1} = \frac{1}{T}\Big\{P[(a+bt_1) - (\sigma a + \sigma bt_1)] + c\Big[-\frac{1}{\theta^2}\Big((-\theta b)e^{\theta(t_1-t_d)} + \theta\big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)}\Big)\big(1+I_e(M-N)\big)\Big] - h\Big[\frac{-1}{\theta^2}\big(\theta\big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)} - \theta be^{\theta(t_1-t_d)}\big)t_d + \frac{1}{\theta^2}(b-\theta a-\theta bt_1) - \big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)} + b\big(e^{\theta(t_1-t_d)}-1\big)\Big] - c\theta\Big[\frac{1}{\theta^2}(b-\theta a - \theta bt_1) - (b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + b\big(e^{\theta(t_1-t_d)}-1\big)\Big] - c[(1-\sigma)(-a-bt_1)] - c\Big[\sigma a(-T+t_1) + \frac{\sigma b}{6}(-6Tt_1 + 4t_1)\Big] - c\Big[-(\sigma a + \sigma bt_1) - \frac{1}{\theta^2}\Big((-\theta b)e^{\theta(t_1-t_d)} + \theta\big(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)}\Big)\Big]\Big\} = 0 \quad (35)$$

The solution to the non-linear equations (34) and (35) gives the value of t_1 and T. To confirm that the optimal solution (t_1^*, T^*) exist and unique, we show that the determinant of the corresponding hessian matrix of the profit function is negative definite $\left(\frac{\partial^2 TP_{12}}{\partial t_1^2}\right) < 0, \left(\frac{\partial^2 TP_{12}}{\partial T^2}\right) < 0$ then

$$\left(\frac{\partial^2 TP_{12}}{\partial T^2}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_{12}}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1 \partial T}\right)|_{(t_1^* T^*)} > 0$$

Using equation (34)

$$\frac{\partial^2 TP_{12}}{\partial T^2} = \frac{1}{\mathrm{T}}\left\{P[\sigma b] - c[\sigma a + \sigma bT] - c[b(1-\sigma)] - c[\sigma b] - 2\frac{\partial TP_{12}}{\partial T}\right\} \quad (36)$$

Evaluating equation (36) at $(t_1^* T^*)$

$$\frac{\partial^2 TP_{11}(t_1^* T^*)}{\partial T^2} < 0 \quad (37)$$

Now using equation (35)

$$\frac{\partial^2 TP_{12}}{\partial t_1^2} = \frac{1}{T}\Big\{P[b-\sigma b] + c\Big[\frac{-1}{\theta^2}\big(\theta^2(b-\theta(a+bt_1)\big)e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})\big(1+I_e(M-N)\big)\Big] - \mathrm{h}\Big[\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1)e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})t_d + \frac{1}{\theta}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)}\Big] - c\theta\Big[\frac{1}{\theta^2}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)} + \theta be^{\theta(t_1-t_d)}\Big] - c[(-b)(1-\sigma)] - c\Big[\sigma a + \frac{\sigma b}{6}(4-6T)\Big] - c\Big[-(\sigma b)\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)} - \theta^2 be^{\theta(t_1-t_d)})\Big]\Big\} \quad (38)$$

Evaluating (38) at $(t_1^* T^*)$

$$\frac{\partial^2 TP_{12}(t_1^* T^*)}{\partial t_1^2} < 0 \quad (39)$$

using equation (34)

$$\frac{\partial^2 TP_{12}}{\partial T\partial t_1} = \frac{1}{T}\{c\sigma[a + bt_1]\} \tag{40}$$

Also, from equation (35)

$$\frac{\partial^2 TP_{12}}{\partial t_1\partial T} = \frac{1}{T}\{c\sigma[a + bt_1]\} \tag{41}$$

Using equations (40) and (41)

$$\left(\frac{\partial^2 TP_{12}}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1\partial T}\right) = \frac{1}{T}\{c\sigma[a + bt_1]\}^2 > 0 \tag{42}$$

Using equations (37), (39) and (42)

$$\left(\frac{\partial^2 TP_{12}}{\partial T^2}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_{12}}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1\partial T}\right) = \Big(-\frac{1}{\mathrm{T}}\{P[\sigma b] - c[\sigma a + \sigma bT] - c[b(1-\sigma)] - c[\sigma b]\}\Big)\Big(\frac{1}{T}\Big\{P[b - \sigma b] + c\Big[\frac{-1}{\theta^2}\big(\theta^2(b - \theta(a + bt_1)\big)e^{\theta(t_1 - t_d)} - \theta^2 be^{\theta(t_1 - t_d)} - \theta^2 be^{\theta(t_1 - t_d)}\big)\big(1 + I_e(M - N)\big)\Big] - \mathrm{h}\Big[\frac{-1}{\theta^2}(\theta^2(b - \theta(a + bt_1)e^{\theta(t_1 - t_d)} - \theta^2 be^{\theta(t_1 - t_d)})t_d + \frac{1}{\theta}(-\theta b) - \theta(b - \theta(a + bt_1))e^{\theta(t_1 - t_d)} + \theta be^{\theta(t_1 - t_d)} + \theta be^{\theta(t_1 - t_d)}\Big] - c\theta\Big[\frac{1}{\theta^2}(-\theta b) - \theta(b - \theta(a + bt_1))e^{\theta(t_1 - t_d)} + \theta be^{\theta(t_1 - t_d)} + \theta be^{\theta(t_1 - t_d)}\Big] - c[(-b)(1-\sigma)] - c\Big[\sigma a + \frac{\sigma b}{6}(4 - 6T)\Big] - c\Big[-(\sigma b)\frac{-1}{\theta^2}(\theta^2(b - \theta(a + bt_1))e^{\theta(t_1 - t_d)} - \theta^2 be^{\theta(t_1 - t_d)} - \theta^2 be^{\theta(t_1 - t_d)})\Big]\Big\} - \frac{1}{T}\{c\sigma[a + bt_1]\}^2 > 0 \tag{43}$$

Since $\left(\frac{\partial^2 TP_{12}}{\partial t_1^2}\right) < 0, \left(\frac{\partial^2 TP_{12}}{\partial T^2}\right) < 0$ and $\left(\frac{\partial^2 TP_{12}}{\partial T^2}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_{12}}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_{12}}{\partial t_1\partial T}\right) > 0$ We conclude that TP_{12} at the point t_1^* and T^* is a maximum. Therefore, equations (36), (38), (40), and (41) satisfied the hessian matrix. then TP_{12} is negative definite.

The necessary conditions for TP_2 to be maximized are $\frac{\partial TP_2}{\partial t_1} = 0, \frac{\partial TP_2}{\partial T} = 0$.

Using equation (31)

$$\frac{\partial TP_2}{\partial T} = \frac{1}{T}\Big\{p[\sigma a + \sigma bT] - C\Big[\sigma a(T - t_1) + \frac{\sigma b}{6}(3T^2 - 3t_1^2)\Big] - c[(1-\sigma)(a + bT)] - c[\sigma a + \sigma bT] - TP_2\Big\} = 0 \tag{44}$$

Also,

$$\frac{\partial TP_2}{\partial t_1} = \frac{1}{T}\Big\{P[(\mathrm{a} + bt_1) - (\sigma a + \sigma bt_1)] - h\Big[\frac{-1}{\theta^2}\big(\theta\big(b - \theta(a + bt_1)\big)e^{\theta(t_1 - t_d)} - \theta be^{\theta(t_1 - t_d)}\big)t_d + \frac{1}{\theta^2}(b - \theta a - \theta bt_1) - \big(b - \theta(a + bt_1)\big)e^{\theta(t_1 - t_d)} + b\big(e^{\theta(t_1 - t_d)} - 1\big)\Big] - c\theta\Big[\frac{1}{\theta^2}(b - \theta a - \theta bt_1) - (b - \theta(a + bt_1))e^{\theta(t_1 - t_d)} + b\big(e^{\theta(t_1 - t_d)} - 1\big)\Big] - c[(1-\sigma)(-a - bt_1)] - c\Big[\sigma a(-T + t_1) + \frac{\sigma b}{6}(-6Tt_1 + 4t_1)\Big] - c\Big[-(\sigma a + \sigma bt_1) - \frac{1}{\theta^2}\Big((-\theta b)e^{\theta(t_1 - t_d)} + \theta\big(b - \theta(a + bt_1)\big)e^{\theta(t_1 - t_d)}\Big)\Big]\Big\} = 0 \tag{45}$$

The solution to the non-linear equations (44) and (45) gives the value of t_1 and T. To confirm that the optimal solution $(t_1^*T^*)$ exist and unique, we show that corresponding hessian matrix $\left(\frac{\partial^2 TP_2}{\partial t_1^2}\right) < 0$, $\left(\frac{\partial^2 TP_2}{\partial T^2}\right) < 0, and$ $\left(\frac{\partial^2 TP\quad_2}{\partial T^2}\right)\left(\frac{\partial^2 TP_2}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_2}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TP_2}{\partial t_1\partial T}\right)|_{(t_1^*T^*)} > 0$

Using (44),

$$\frac{\partial^2 TP_2}{\partial T^2} = \frac{1}{\mathrm{T}}\Big\{P[\sigma b] - c[\sigma a + \sigma bT] - c[b(1-\sigma)] - c[\sigma b] - 2\Big\{\frac{\partial TP_2}{\partial T}\Big\}\Big\} \tag{46}$$

Evaluating equation (46) at $(t_1^*T^*)$

$$\frac{\partial^2 TP_2}{\partial T^2}|_{(t_1^*T^*)} < 0 \tag{47}$$

Using equation (45),

$$\frac{\partial^2 TP_2}{\partial t_1^2} = \frac{1}{T}\Big\{P[b-\sigma b] - \mathrm{h}\left[\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1)e^{\theta(t_1-t_d)} - \theta^2 b e^{\theta(t_1-t_d)})t_d + \frac{1}{\theta}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)}\right] - c\theta\left[\frac{1}{\theta^2}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)}\right] - c[(-b)(1-\sigma)] - c\left[\sigma a + \frac{\sigma b}{6}(4-6T)\right] - c\left[-(\sigma b)\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 b e^{\theta(t_1-t_d)} - \theta^2 b e^{\theta(t_1-t_d)})\right]\Big\} \quad (48)$$

Evaluating equation (48) at $(t_1^* T^*)$

$$\frac{\partial^2 TP_2(t_1^* T^*)}{\partial t_1^2} < 0 \quad (49)$$

Using equation (44)

$$\frac{\partial^2 TP_2}{\partial T \partial t_1} = \frac{1}{T}\{c\sigma[a+bt_1]\} \text{ since } \frac{\partial TP_2}{\partial t_1} = 0 \quad (50)$$

Also, from equation (45)

$$\frac{\partial^2 TP_2}{\partial t_1 \partial T} = \frac{1}{T}\{c\sigma[a+bt_1]\} \quad (51)$$

$$\left(\frac{\partial^2 TP_2}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TP_2}{\partial t_1 \partial T}\right) = \frac{1}{T}\{c\sigma[a+bt_1]\}^2 > 0 \quad (52)$$

Using equations (47), (49), (50) and (51)

$$\left(\frac{\partial^2 TP_2}{\partial T^2}\right)\left(\frac{\partial^2 TP_2}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_2}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TP_2}{\partial t_1 \partial T}\right) = \Big(-\frac{1}{\mathrm{T}}\{P[\sigma b] - c[\sigma a + \sigma bT] - c[b(1-\sigma)] - c[\sigma b]\Big)\Big(\frac{1}{T}\Big\{P[b-\sigma b] - \mathrm{h}\left[\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1)e^{\theta(t_1-t_d)} - \theta^2 b e^{\theta(t_1-t_d)})t_d + \frac{1}{\theta}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)}\right] - c\theta\left[\frac{1}{\theta^2}(-\theta b) - \theta(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)} + \theta b e^{\theta(t_1-t_d)}\right] - c[(-b)(1-\sigma)] - c\left[\sigma a + \frac{\sigma b}{6}(4-6T)\right] - c\left[-(\sigma b)\frac{-1}{\theta^2}(\theta^2(b-\theta(a+bt_1))e^{\theta(t_1-t_d)} - \theta^2 b e^{\theta(t_1-t_d)} - \theta^2 b e^{\theta(t_1-t_d)})\right]\Big\}\Big) - \left(\frac{1}{T}\{c\sigma[a+bt_1]\}^2\right) < 0$$
(53)

Since $\left(\frac{\partial^2 TP_2}{\partial t_1^2}\right) < 0, \left(\frac{\partial^2 TP_2}{\partial T^2}\right) < 0$ *and* $\left(\frac{\partial^2 TP_2}{\partial T^2}\right)\left(\frac{\partial^2 TP_2}{\partial t_1^2}\right) - \left(\frac{\partial^2 TP_2}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TP_2}{\partial t_1 \partial T}\right) > 0$ We conclude that TP_2 at the point t_1^* and T^* is a maximum. Therefore, equation (47), (49), (50) and (51) satisfied the hessian matrix. Therefore, TP_2 is negative definite.

RESULTS AND DISCUSSION

Consider an inventory system with the following input parameters. A=150; a=200; b=3; P=150; h=10; M=40; N=35; c=10; Ie=0.4; t_d=25; σ=0.8; r = θ=0.2 and t_d=30 for Case 2.

Using MAPLE software based on the parameter values provided for case 1 (scenario 1), it was found that $\boldsymbol{t_1^{11*}}$ =24.5070, $\boldsymbol{T_{11}^*}$= 31.7335 and $\boldsymbol{TP_{11}^*}$= \$16000, which showed that offering customers with trade credit and not accepting partial payments can lead to higher profitability. The result for all the cases is provided below:

Table 1: Results for the numerical Analysis for all cases.

CASES	t_1	T	TP (in dollars)
Case 1 (scenario 1)	25.2374	36.2394	8200
Case 1 (scenario 2)	24.5070	31.7335	16000
Case 2	29.1121	40.6600	7200

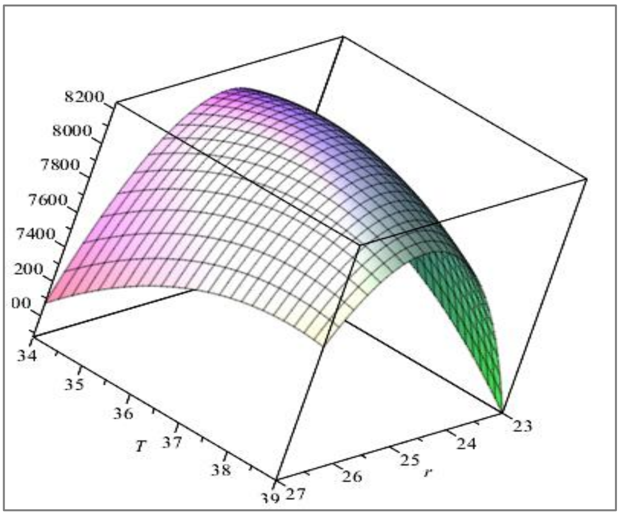

Figure 2: Graphical Representation of the Numerical Analysis for Case 1 (scenario 1).

The result shows that when a partial payment is not acceptable at the expiration of initial agreed permissible delay in payment period, N, but complete payment later at M>N, presents the optimal situation with $\boldsymbol{t_1}$ =

$\mathbf{24.5070}$, $\boldsymbol{T} = 31.7335$ and a profit of \$16000. This implies that the retailer should always ensure items are fresh to achieve higher profits.

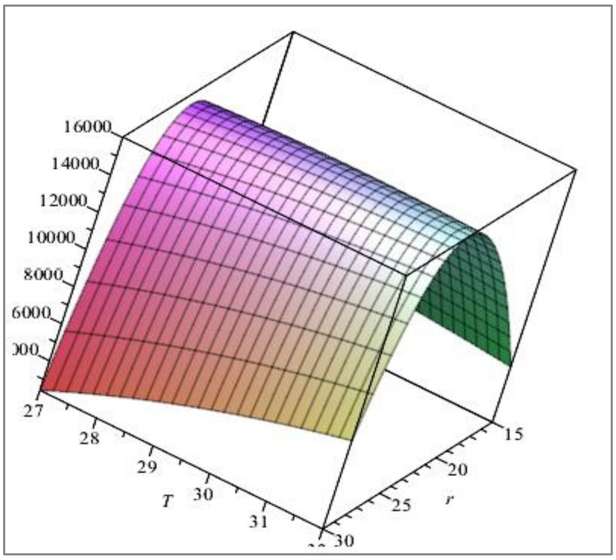

Figure 3: Graphical Representation of the Numerical Analysis for Case 1 (scenario 2).

This is because the optimal case is when a large chunk of goods has been sold before deterioration sets in, and the cycle ends immediately once items begin to deteriorate. Meaning less inventory in the store during the period. The results in Table 1 are presented in Figures 2, 3, and 4.

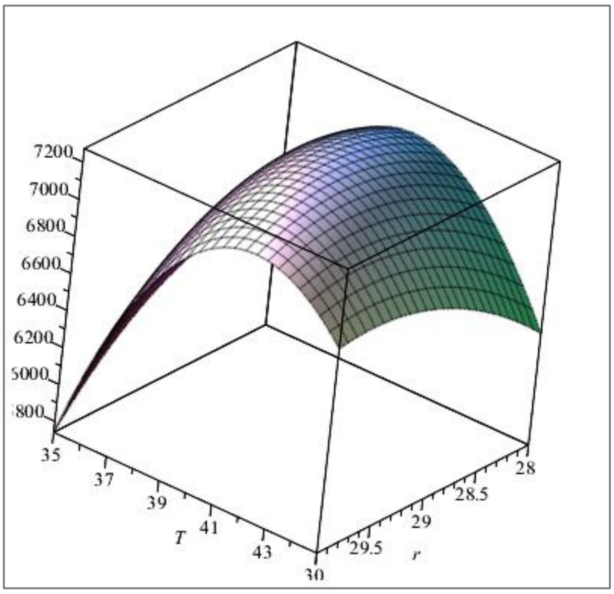

Figure 4: Graphical Representation of the Numerical Analysis for Case 2.

SENSITIVITY ANALYSIS

The sensitivity is performed on some parameters A, h, N, σ, and θ. The result is represented in Table 2, 3 and 4.

Table 2: Sensitivity analysis result of case 1(Scenario 1)

Parameter	% change in parameter	New t_1^*	% in t_1^*	New T^*	% in T^*	New TP_{11}^*	% in TP_{11}^*
	+20	25.2374	0	36.2399	0.0014	8200	0
	+10	25.2374	0	36.2396	0.0006	8200	0
A	+0	25.2374	0	36.2394	0	8200	0
	-10	25.2373	0.0004	36.2392	-0.006	8200	0
	-20	25.2373	0.0004	36.2389	-0.0014	8200	0
	+20	24.7162	-2.0652	37.6661	3.9369	4000	-51.2195
	+10	24.9598	-1.1100	36.9564	1.9785	6000	-26.8293
h	+0	25.2374	0	36.2394	0	8200	0
	-10	25.5583	1.2715	35.5192	-1.9873	10.400	26.8293
	-20	25.9362	2.7689	34.8023	-3.9655	12,500	52.4900
	+20	24.3860	-3.3736	34.3078	-5.3301	10400	26.8293
	+10	24.8447	-1.5560	35.5324	-1.9509	8800	7.3170
N	+0	25.2374	0	36.2394	0	8200	0
	-10	25.5805	1.3595	36.4997	0.7183	8400	2.4390
	-20	25.8830	2.5581	36.3514	0.3091	9400	14.6341
	+20	25.3554	0.4676	36.4463	0.5709	10200	24.3902
	+10	25.2963	0.2337	36.3468	0.2964	9200	12.1951
σ	+0	25.2374	0	36.2394	0	8200	0
	-10	25.1787	-0.2326	36.1214	-0.3256	7200	-12.1951
	-20	25.1203	-0.4640	35.9884	-0.6929	6000	-26.8293
	+20	25.0214	-0.8559	36.0178	-0.6115	8200	0
	+10	25.1118	-0.4977	36.1114	-0.3532	8200	0
θ	+0	25.2374	0	36.2394	0	8200	0
	-10	25.4158	0.7069	36.4179	0.4923	8200	0
	-20	25.6763	1.7395	36.6732	1.1970	8200	0

Table 3: Sensitivity analysis result of case 1 (scenario 2)

Parameter	% change in parameter	New t_1^*	% in t_1^*	New T^*	% in T^*	New TP_{12}^*	% in TP_{12}^*
	+20	24.5071	0.0004	31.7341	0.0019	16000	0
	+10	24.5071	0.0004	31.7337	0.0006	16000	0
A	+0	24.5071	0	31.7335	0	16000	0
	-10	24.5070	0	31.7332	-0.0009	16000	0
	-20	24.5070	-0.0004	31.7329	-0.0019	16000	0
	+20	24.1093	-1.6228	33.5389	5.6893	10000	-37.5
	+10	24.2962	-0.8602	32.6502	2.8887	12000	-25
h	+0	24.5070	0	31.7335	0	16000	0
	-10	24.7498	0.9862	30.7888	-2.9770	18000	12.5
	-20	25.0314	2.1398	29.8178	-6.0368	20000	25
	+20	24.3266	-0.7361	33.9358	6.9400	10,000	-37.5
	+10	24.2662	-0.3648	32.8650	3.5656	12000	-25
N	+0	24.5070	0	31.7335	0	16000	0
	-10	24.7487	0.3501	30.5307	-3.7903	18000	12.5
	-20	25.6720	0.6733	29.2428	-7.8488	20000	25
	+20	24.5458	0.5663	36.6494	2.8862	16000	0
	+10	24.5769	0.2852	32.2301	1.5649	16000	0
σ	0	24.5070	0	31.7335	0	16000	0
	-10	24.4356	-0.2913	31.1309	-1.8989	14000	-12.5
	-20	24.3618	-0.5925	30.3774	-4.2734	14000	-12.5
	+20	24.3909	-0.4737	31.5884	-0.4572	16000	0
	+10	24.4347	-0.2950	31.6447	-0.2798	16000	0
θ	0	24.5070	0	31.7335	0	16000	0
	-10	24.6244	0.4790	31.8722	0.4371	16000	0
	-20	24.8150	1.2568	32.0890	1.1203	15000	-6.25

Table 4: Sensitivity analysis result of case 2

Parameter	% change in parameter	New t_1^*	% in t_1^*	New T^*	% in T^*	New TP_2^*	% in TP_2^*
	+20	29.1122	0.0003	40.6604	0.0010	7200	0
	+10	29.1121	0	40.6602	0.8132	7200	0
A	0	29.1121	0	40.6600	0	7200	0
	-10	29.1121	0	40.6598	-0.0005	7200	0
	-20	29.1121	0	40.6596	-0.0010	7200	0
	+20	28.7201	-1.3465	42.5954	4.7600	1500	-79.1667
	+10	28.9053	-0.7104	41.6405	2.4122	4500	-37.5
h	+0	29.1121	0	40.6600	0	7200	0
	-10	29.3459	0.8031	39.6529	-2.4768	10000	38.8889
	-20	29.6154	1.7247	38.6204	-5.0162	12000	77.7778
	+20	29.1121	0	40.6600	0	7200	0
	+10	29.1121	0	40.6600	0	7200	0
N	+0	29.1121	0	40.6600	0	7200	0
	-10	29.1121	0	40.6600	0	7200	0
	-20	29.1121	0	40.6600	0	7200	0
	+20	29.2750	0.5596	40.7765	0.2865	9400	30.556
	+10	29.1936	0.2800	40.7164	0.387	8200	13.8889
σ	0	29.1121	0	40.6600	0	7200	0
	-10	29.0305	-0.2803	40.6084	-0.1269	6000	-16.6667
	-20	28.9487	-0.5613	40.5638	-0.2366	5000	-30.5556
	+20	29.1160	0.0134	40.6500	-0.0246	7200	0
	+10	29.1076	-0.0154	40.6471	-0.0317	7200	0
θ	0	29.1121	0	40.6600	0	7200	0
	-10	29.1394	0.0938	40.6993	0.0966	7200	0
	-20	29.2068	0.3253	40.7838	0.3045	7200	0

DISCUSSION OF RESULTS

It is observed from Tables 2, 3, and 4 that:

1. As the ordering cost A increases t_1^* , T* increase while TP* remains unchanged for all cases. In real life, when ordering is high, the retailer tends to order more goods, which leads to an an increase in the replenishment cycle time and total profit. But the rise in holding costs offsets fewer orders, keeping the total profit unchanged.

2. As the holding cost h increases t_1^* decreases, T* increases, and TP* decreases for all cases. In real life, when holding costs rise, retailers tend to order smaller batches, leading to shorter cycle lengths and shorter cycle times, but also to higher inventory costs, which may cause a fall in profit.
3. As the downstream credit period N increases, t_1^*, $T * decreases$, T* while TP* increases for case 1 (scenario 1). Then, t_1^* decrease, T* increases, and TP* decreases for case 1 (scenario 2). In real life, an extended trade credit period can decrease cycle time and increase sales volume, leading to higher total profit. But sometimes, the cost associated with higher credit periods can reduce the business total profit.
4. As the backlogging rate σ increases, t_1^*, T* TP* increases for all cases. This aligns with the expectation of real life. When the backlogging rate is high, it can lead to an increase in the replenishment cycle time and the total profit.
5. As the deterioration rate θ increases, t_1^*, T* decreases while TP* remain unchanged for all cases. In real-life scenarios, when deterioration is high, the cycle time and the replenishment length actually decrease. But since sales remain constant and demand fulfilment is unaffected, profit stays unchanged.

The model effectively captures both operational and financial dynamics by incorporating trade credit, delayed deterioration, and customer behavior during shortages. This study presents an inventory model for non-instantaneous deterioration items under a retailer's trade credit with partial backlogging. This model builds on the work of Vadana and Sharma (2016) and Malumfashi et al. (2023) by incorporating the retailer's trade credit and partial backlogging for non-instantaneous, deteriorating items. Based on the numerical examples provided, case 1 (scenario 1) it was found that $\boldsymbol{t_1^{11*}}$ =24.5070, $\boldsymbol{T_{11}^*}$= 31.7335 and $\boldsymbol{TP_{11}^*}$=\$16000 is the optimal policy with optimal values of the total profit and the decision variables.

CONCLUSION AND RECOMMENDATION

For managerial implications, the model shows that it is more cost-effective for the retailer to order items with a longer period of maintaining freshness to earn maximum profit. This is because the optimal case is when a large portion of goods has been sold before deterioration sets in, and the cycle ends immediately once items begin to deteriorate, which means less inventory in the store during the deterioration period. This model offers a more realistic framework that reflects actual market conditions, where trade credit and shortages do not always lead to a loss of sales.

For future research, this can be extended to include multiple items, stochastic demand or multi-echelon supply chains. Also, for a more comprehensive analysis, inflation and discount can be incorporated. Furthermore, the model can be extended to include dynamic pricing strategies under a trade credit policy.

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