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ORIGINAL RESEARCH ARTICLE

Quasi-Convolution Properties of Analytic Function Subclasses Defined by a Generalized Binomial Operator

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ABSTRACT

The study of analytic and univalent functions in the open unit disk has received significant attention in geometric function theory due to its vast applications across mathematics and other related disciplines. Operators, being one of the essential tools, preserve geometric properties and also help generate various subclasses of analytic and univalent functions. This study introduces a new subclass of analytic and univalent functions defined via a novel generalized differential operator, extending the Al-Oboudi and Bello-Babalola operator through a binomial series formulation. The analysis is based on the close relationship between the proposed class and the class of functions with positive real part in the unit disk. The proofs employ standard techniques from classical geometric function theory, supported by established lemmas from the literature. In addition, convolution (Hadamard product) methods are utilized to derive inclusion properties and related characterizations of the class. Several fundamental properties are established, including inclusion relationships and key structural characteristics. In particular, convolution closure properties are derived, alongside coefficient bounds related to partial sums. Connections between the proposed subclass and previously studied classes defined by differential operators, such as the Sălăgean, Al-Oboudi, and Bello-Babalola operators, are also identified. The study provides new insights into the structure and behavior of analytic and univalent function subclasses.

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INTRODUCTION

Let $q \in \mathbb{N} = \{1,2,3, \ldots\}$, and let $\mathcal{A}$ denote the class analytic functions $f(z)$ normalized by $(0) = 0$ and $f'(0) = 1$, having the series representation,

$$f(z) = z + \sum_{q=2}^{\infty} a_q z^q, \qquad z \in \mathbb{E}, \qquad (1)$$

in the open disk,

$$\mathbb{E} = \{z \in \mathbb{C}: |z| = r < 1\},$$

where $\mathbb{N}$ and $\mathbb{C}$ represent the set of natural and complex numbers respectively. Subsequent references of $f(z)$ or z are as defined above (Babalola, 2008; Babalola, 2016).

Caratheodory functions: Let P denote the class of functions

$$P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

which are analytic in unit disk $\mathbb{E}$ and satisfy

$$Rep(z) > 0, \qquad z \in \mathbb{E}$$

(Olatunji *et al.*,2024).

Recently, considerable attention has been devoted to the study of subclasses of analytic functions defined using differential operators, since these operators serve as important tools for generating, preserving, and characterizing function classes with desirable geometric properties. Many researchers have introduced various operators and employed them to investigate geometric properties of different classes of analytic and univalent functions. Rossdy et al. (2024) introduced an operator associated with binomial series by combining the Srivastava–Attiya and Al-Oboudi operators, and employed it to study subclasses of analytic functions using the method of differential subordination. Further developments on differential operators and related subclasses of analytic function are provided by Al-Oboudi (2004), Srivastava and Attiya (2007), Bello (2023), Rossdy et al. (2024), and Byeon et al. (2024).

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Several related studies have also contributed to the advancement of generalized mathematical operators, asymptotic analysis, and analytical techniques associated with differential systems and functional structures. For instance, Abba et al. (2024) investigated combinatorial structures associated with permutation pattern avoidance, while Abdullahi et al. (2024) studied asymptotic properties of generalized statistical distributions. Bello and Mutawakilu (2024) developed polynomial series approaches for nonlinear boundary value problems, whereas Balogun et al. (2024) explored generalized combinatorial frameworks involving partially ordered multisets. Furthermore, convergence and stability properties of numerical and differential analytical methods have been investigated by Adee and Kumleng (2022), Bala and Musa (2022), and Alhassan et al. (2024). Collectively, these studies demonstrate the continuing development of generalized mathematical frameworks and analytical operators in modern mathematical research, thereby motivating further investigations into generalized differential operators and their associated subclasses of analytic and univalent functions.

Convolution (or Hadamard product) is another important concept in this area, as it provides a mechanism for combining analytic functions while preserving geometric properties. Investigations of convolution properties have led to significant insights into inclusion relationships, coefficient estimates, and several other characteristics of function classes. The concept of quasi-convolution, which extends the traditional convolution, was first introduced by Babalola in 2008. Since then, several researchers have continued to contribute to this area (Srivastava and Attiya, 2007; Lasode and Opoola, 2023).

Let

$$g(z) = z + \sum_{q=2}^{\infty} b_q z^q,$$

$f, g \in \mathcal{A}$, and $q \in \mathbb{N}$. Then the convolution (or Hadamard product) of $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{q=2}^{\infty} a_q b_q z^q, \quad (z \in \mathbb{E}) \qquad (2)$$

(Owa and Strivastava, 2002; Zhongzhu and Owa, 1992; Kang, Owa and Strivastava, 1996; Lasode and Opoola, 2023).

Now, for $\rho > o, z \in \mathbb{E}$ and $k \in \mathbb{N}$ we may write $f(z)^\rho$ and $g(z)^\rho$ as

$$f(z)^\rho = z^\rho + \sum_{k=2}^{\infty} A_k(\rho) z^{\rho+k-1},$$

and

$$g(z)^\rho = z^\rho + \sum_{k=2}^{\infty} \mathcal{B}_k(\rho) z^{\rho+k-1}$$

where $A_k(\rho)$ and $\mathcal{B}_k(\rho)$ depend respectively on the coefficients a_q of $f(z)$ and b_q of $g(z)$, and the parameter ρ (Babalola, 2008; Bello and Babalola, 2025).

The Quasi-convolution of $f(z)^\rho$ and $g(z)^\rho$ is defined as follows (Babalola, 2008):

$$f(z)^\rho * g(z)^\rho = z^\rho + \sum_{k=2}^{\infty} A_k(\rho)\mathcal{B}_k(\rho) z^{\rho+k-1}, \quad \rho > o, z \in \mathbb{E} \qquad (3)$$

The binomial series is defined by

$$(1-\vartheta)^v = \sum_{v=0}^{d} \binom{d}{v} (-1)^v (-\vartheta)^v, \quad (d \in \mathbb{N}, v \in \mathbb{N}_0 = \{0,1,2,3,\ldots\}, 0 < \vartheta < 1)$$

(Rossdy *et al.*, 2024).

The Al-Oboudi differential operator is defined as follows (Al-Oboudi, 2004):

$$D_\zeta^0 f(z) = f(z)$$

$$D_\zeta^1 f(z) = (1-\zeta) f(z) + \zeta z [f(z)]', \quad \zeta \geq 0, z \in \mathbb{E}$$

And recursively,

$$D_\zeta^q f(z) = (1-\zeta) D_\zeta^{q-1} f(z) + \zeta z \left[D_\zeta^{q-1} f(z)\right]', \quad q \in \mathbb{N}.$$

Motivated by these developments, this paper presents a new subclass of analytic and univalent functions defined by a generalized differential operator using convolution techniques.

MATERIALS AND METHODS

The Generalized Differential Operator

Definition 2.1 For $f \in \mathcal{A}$, $z \in \mathbb{E}, \rho > 0$, $\zeta \geq 0$, $0 < \vartheta < 1$, $v \in \mathbb{N}$, and $q \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, we introduce the generalized operator as follows:

$$\mathfrak{B}_{\zeta,\rho,v}^0 f(z)^\rho = f(z)^\rho,$$

$$\mathfrak{B}_{\zeta,\rho,v}^1 f(z)^\rho = \rho(1-\zeta(1-\vartheta)^v) f(z)^\rho + \zeta(1-\vartheta)^v z [f(z)^\rho]',$$

And recursively,

$$\mathfrak{B}_{\zeta,\rho,v}^q f(z)^\rho = \rho(1-\zeta(1-\vartheta)^v) \mathfrak{B}_{\zeta,\rho,v}^{q-1} f(z)^\rho + \zeta(1-\vartheta)^v z \left[\mathfrak{B}_{\zeta,\rho,v}^{q-1} f(z)^\rho\right]'. \qquad (4)$$

The series form of the generalized operator is given by

$$\mathfrak{B}_{\zeta,\rho,v}^q f(z)^\rho = \rho^q z^\rho + \sum_{k=2}^{\infty} [\rho + \zeta(1-\vartheta)^v (k-1)]^q A_k(\rho) z^{\rho+k-1}. \qquad (5)$$

Remark 2.1 The operator $\mathfrak{B}^{q}_{\zeta,\rho,v}f(z)^{\rho}$ generalizes several previously known operators.
In particular:
When $v = 0$, it reduces to $\mathfrak{B}^{q}_{\zeta,\rho,0}f(z)^{\rho} = D^{q}_{\zeta,\rho}f(z)^{\rho}$, which was studied by Bello and Babalola (2025).
When $\rho = 1$ and $v = 0$, it reduces to $\mathfrak{B}^{q}_{\zeta,1,0}f(z) = D^{q}_{\zeta}f(z)$, which was studied by Al-Oboudi (2004).
When $\rho = \zeta = 1$ and $v = o$, it reduces to the Sălăgean differential operator (Babalola, 2008; Babalola, 2016).
Example 2.1 Consider the analytic function.

$$f(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^{k}, \qquad k \in \mathbb{N}, \qquad z \in \mathbb{E},$$

Setting $\rho = 1$, we obtain

$$f(z)^{\rho} = f(z).$$

Using Definition 1.1 with $\rho = 1$, the generalized operator becomes

$$\mathfrak{B}^{1}_{\zeta,1,v}f(z) = (1-\zeta(1-\vartheta)^{v})f(z) + \zeta(1-\vartheta)^{v}zf'(z).$$

We have

$$\mathfrak{B}^{1}_{\zeta,1,v}f(z) = (1-\zeta(1-\vartheta)^{v})\left[\sum_{k=1}^{\infty} z^{k}\right] + \zeta(1-\vartheta)^{v}z\left[\sum_{k=1}^{\infty} z^{k}\right]'.$$

That is,

$$\mathfrak{B}^{1}_{\zeta,1,v}f(z) = (1-\zeta(1-\vartheta)^{v})\left[\sum_{k=1}^{\infty} z^{k}\right] + \zeta(1-\vartheta)^{v}\left[\sum_{k=1}^{\infty} kz^{k}\right].$$

After simplification,

$$\mathfrak{B}^{1}_{\zeta,1,v}f(z) = \sum_{k=1}^{\infty}[1+\zeta(1-\vartheta)^{v}(k-1)]z^{k}.$$

This example illustrates how the generalized operator transforms a standard analytic function into another analytic function depending on the parameters ζ, ϑ and v.
Remark 2.2 Setting $v = 0$, we obtain

$$\mathfrak{B}^{1}_{\zeta,1,0}f(z) = \sum_{k=1}^{\infty}[1+\zeta(k-1)]z^{k}.$$

This is equivalent to the Al-Oboudi operator, namely,

$$\mathfrak{B}^{1}_{\zeta,1,0}f(z) = D^{1}_{\zeta}f(z).$$

Example 2.2 Consider the analytic function

$$f(z) = \left(\frac{z^{\rho}}{1-z}\right)^{\frac{1}{\rho}}, \qquad k \in \mathbb{N}, \qquad z \in \mathbb{E}.$$

Equivalently,

$$f(z)^{\rho} = \frac{z^{\rho}}{1-z} = z^{\rho} + \sum_{k=2}^{\infty} z^{\rho+k-1}.$$

Using Definition 1.1, the generalized operator becomes

$$\mathfrak{B}^{1}_{\zeta,\rho,v}f(z)^{\rho} = \rho(1-\zeta(1-\vartheta)^{v})\left[z^{\rho} + \sum_{k=2}^{\infty} z^{\rho+k-1}\right] + \zeta(1-\vartheta)^{v}z\left[z^{\rho} + \sum_{k=2}^{\infty} z^{\rho+k-1}\right]'.$$

After simplification, we obtain

$$\mathfrak{B}^{1}_{\zeta,\rho,v}f(z)^{\rho} = \rho z^{\rho} + \sum_{k=2}^{\infty}[\rho+\zeta(1-\vartheta)^{v}(k-1)]z^{\rho+k-1}.$$

Remark 2.3 Setting $\rho = 1$, we obtain

$$\mathfrak{B}^{1}_{\zeta,1,v}f(z) = z + \sum_{k=2}^{\infty}[1+\zeta(1-\vartheta)^{v}(k-1)]z^{k}.$$

This coincides with the result obtained in Example 2.1. Hence,

$$\mathfrak{B}^{1}_{\zeta,1,v}f(z) = \sum_{k=1}^{\infty}[1+\zeta(1-\vartheta)^{v}(k-1)]z^{k}.$$

The New Subclass: $\mathcal{K}_{q}(\rho, \zeta, \xi, v)$

Definition 2.2 A function $f(z)$ belongs to the class $\mathcal{K}_q(\rho,\zeta,\xi,v)$ if and only if it satisfies the inequality

$$Re\left(\frac{\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho}{\rho^q z^\rho}\right) > \xi, \tag{6}$$

for $f \in \mathcal{A}$, $z \in \mathbb{E}, \rho > 0, 0 \le \xi < 1, \zeta \ge 0$, $0 < \vartheta < 1$, $v \in \mathbb{N}$, and $q \in \mathbb{N}_0$.

Remark 2.4 The class $\mathcal{K}_q(\rho,\zeta,\xi)$ generalizes several existing subclasses in the literature**.**

For instance, when $v = 0$, the class reduces to $\mathcal{K}_q(\rho,\zeta,\xi)$, studied by Bello and Babalola (2025).

When $v = 0$ and $\zeta = 1$, it reduces to the class $\mathcal{K}_q(\rho,1,\xi)$, studied by Babalola (2008).

When $v = 0, n = 0$ and $\zeta = 1$, it reduces to the class $\mathcal{K}_0(\rho,1,\xi)$, studied by Abdulhalim (2003).

The following lemmas shall be needed to proof our theorems.

Lemma 2.1

If $p(z)$ is analytic in $\mathbb{E}$, satisfies $p(0) = 1$, and

$$Rep(z) > \frac{1}{2}, \qquad z \in \mathbb{E},$$

then, for any function $h(z)$ analytic in $\mathbb{E}$, the convolution $p * h$ takes its values in the convex hull of $h(\mathbb{E})$ (Babalola, 2008).

Lemma 2.2

Let $\{u\}_{k=0}^{\infty}$ be a convex null sequence. Then the function

$$p(z) = \frac{u_0}{2} + \sum_{k=1}^{\infty} u_k z^k, \quad k \in \mathbb{N}, \qquad z \in \mathbb{E}$$

is analytic in $\mathbb{E}$, and satisfies $Rep(z) > 0$ (Babalola, 2008).

Lemma 2.3

Let $l \in \mathbb{N}$ and $-1 < t < s = 4.567802$. Then

$$Re\left(\sum_{k=2}^{l} \frac{z^{k-1}}{t+k-1}\right) > -\frac{1}{1+t}, \qquad k \in \mathbb{N}, \qquad z \in \mathbb{E}.$$

(Al-Oboudi, 2004.)

Lemma 2.4 Let $\delta > 0$ and $T > 0$. If $p(z)$ is analytic in $\mathbb{E}$ satisfies $p(0) = 1$ and

$$|p(z) + \delta z p'(z) - 1| < T,$$

then

$$|p(z) - 1| < \frac{T}{1+\delta}.$$

(Zhongzhu and Owa, 1992)

RESULTS AND DISCUSSION

Theorem 3.1

Let $0 \le \xi < 1$ and $f \in \mathcal{K}_q(\rho,\zeta,\xi,v)$. If $\left|\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q+1}z^\rho} - 1\right| < 1 - \xi$, then $\left|\frac{\mathfrak{B}^{q}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q}z^\rho} - 1\right| < \frac{(1-\xi)\rho}{\rho+\zeta(1-\vartheta)^v}$.

Proof For $f \in \mathcal{K}_q(\rho,\zeta,\xi,v)$, let

$$\frac{\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho}{\rho^q z^\rho} = p(z). \tag{7}$$

Equivalently,

$$\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho = \rho^q z^\rho p(z). \tag{8}$$

Using the recursive definition of the operator, we have

$$\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho = \rho(1-\zeta(1-\vartheta)^v)\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho + \zeta(1-\vartheta)^v z\left[\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho\right]'. \tag{9}$$

Substituting (8) into (9), we obtain

$$\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho = (1-\zeta(1-\vartheta)^v)\rho^{q+1}z^\rho p(z) + \zeta(1-\vartheta)^v z[\rho^q z^\rho p(z)]'.$$

Differentiating $\rho^q z^\rho p(z)$, we have

$$\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho = (1-\zeta(1-\vartheta)^v)\rho^{q+1}z^\rho p(z) + \zeta(1-\vartheta)^v z\left[\rho^{q+1}z^{\rho-1}p(z) + \rho^q z^\rho p'^{(z)}\right].$$

Further simplification gives,

$$\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho = \rho^{q+1}z^\rho p(z) + \zeta(1-\vartheta)^v \rho^q z^{\rho+1} p'(z). \tag{10}$$

Dividing both sides of (10) by $\rho^{q+1}z^\rho$, we obtain

$$\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q+1}z^\rho} = p(z) + \frac{\zeta(1-\vartheta)^v}{\rho} z p'(z). \tag{11}$$

It follows that,

$$\left|\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q+1}z^\rho}-1\right| = \left|p(z)+\frac{\zeta(1-\vartheta)^v}{\rho}zp'(z)-1\right|. \tag{12}$$

By hypothesis,

$$\left|\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q+1}z^\rho}-1\right| < 1-\xi, \tag{13}$$

Applying Lemma 2.4 with $\delta=\frac{\zeta(1-\vartheta)^v}{\rho}$ and $T=1-\xi$, we obtain

$$\left|\frac{\mathfrak{B}^{q}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q}z^\rho}-1\right| < \frac{1-\xi}{1+\frac{\zeta(1-\vartheta)^v}{\rho}}, \tag{14}$$

Therefore,

$$\left|\frac{\mathfrak{B}^{q}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q}z^\rho}-1\right| < \frac{(1-\xi)\rho}{\rho+\zeta(1-\vartheta)^v},$$

Hence, the proof is established.

Corollary 3.1 Let $0\le\xi<1$ and $f\in\mathcal{K}_q(\rho,\zeta,\xi,0)$. If $\left|\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,0}f(z)^\rho}{\rho^{q+1}z^\rho}-1\right|<1-\xi$, then

$$\left|\frac{\mathfrak{B}^{q}_{\zeta,\rho,0}f(z)^\rho}{\rho^{q}z^\rho}-1\right| < \frac{(1-\xi)\rho}{\rho+\zeta}.$$

Corollary 3.2 Let $0\le\xi<1$ and $f\in\mathcal{K}_q(1,\zeta,\xi,0)$. If $\left|\frac{\mathfrak{B}^{q+1}_{\zeta,1,0}f(z)}{z}-1\right|<1-\xi$, then

$$\left|\frac{\mathfrak{B}^{q}_{\zeta,1,0}f(z)}{z}-1\right| < \frac{(1-\xi)}{1+\zeta}.$$

Remark 3.1 Setting $T=1-\xi$ in Corollary 3.2, we obtain the result obtained by Zhongzhu and Owa, (1992).

Theorem 3.2

Let $f\in\mathcal{K}_q(\rho,\zeta,\xi,v)$. Then $\mathcal{K}_{q+1}(\rho,\zeta,\xi,v)\subset\mathcal{K}_q(\rho,\zeta,\xi,v)$.

Proof For $f\in\mathcal{K}_{q+1}(\rho,\zeta,\xi,v)$, from (5) we have

$$\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho=\rho^{q+1}z^\rho + \sum_{k=2}^{\infty}[\rho+\zeta(1-\vartheta)^v(k-1)]^{q+1}A_k(\rho)z^{\rho+k-1}. \tag{15}$$

Consequently,

$$\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q+1}z^\rho}=1+\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q+1}z^{k-1}. \tag{16}$$

Since $f\in\mathcal{K}_{q+1}(\rho,\zeta,\xi,v)$, it follows that

$$Re\,\frac{\mathfrak{B}^{q+1}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q+1}z^\rho}=Re\left[1+\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q+1}z^{k-1}\right]>\xi. \tag{17}$$

Equivalently,

$$Re\left[1-\xi+\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q+1}z^{k-1}\right]>0. \tag{18}$$

Dividing (18) by $1-\xi$ and adding $+1$ to both sides, we have

$$Re\left[1+\frac{1}{2(1-\xi)}\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q+1}z^{k-1}\right]>\frac{1}{2}. \tag{19}$$

Applying the definition of convolution, we have

$$\frac{\mathfrak{B}^{q}_{\zeta,\rho,v}f(z)^\rho}{\rho^{q}z^\rho}=1+\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q}z^{k-1},$$

which may be written as

$$\frac{\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho}{\rho^q z^\rho} = \left(1+\frac{1}{2(1-\xi)}\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q+1}z^{k-1}\right)$$
$$*\left(1+2(1-\xi)\sum_{k=2}^{\infty}\frac{\rho z^{k-1}}{\rho+(k-1)\zeta(1-\vartheta)^v}\right). \tag{20}$$

In view of Lemma 2.2, taking $u_0 = 1,\ u_k = \frac{\rho}{\rho+k\zeta(1-\vartheta)^v}, k \in N$, and $Rep(z) > 0$, we have

$$Re\left(\frac{1}{2}+\sum_{k=2}^{\infty}\frac{\rho z^{k-1}}{\rho+(k-1)\zeta(1-\vartheta)^v}\right) > 0. \tag{21}$$

Multiplying (21) by $2(1-\xi)$ and simplifying gives

$$Re\left(1+2(1-\xi)\sum_{k=2}^{\infty}\frac{\rho z^{k-1}}{\rho+(k-1)\zeta(1-\vartheta)^v}\right) > \xi. \tag{22}$$

By Lemma 2.1 and (20), we conclude that

$$Re\left(\frac{\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho}{\rho^q z^\rho}\right) > \xi.$$

Therefore, $f \in \mathcal{K}_q(\rho,\zeta,\xi,v)$, and consequently,

$$\mathcal{K}_{q+1}(\rho,\zeta,\xi,v) \subset \mathcal{K}_q(\rho,\zeta,\xi,v).$$

Remark 3.2 Setting $v = 0$ in Theorem 3.2, we obtain the result established by Bello and Babalola (2025).

Theorem 3.3

Let $l \in \mathbb{N}$, and let $\mathcal{S}_l(z,f)$ be the l-th partial sum of a function of $f \in \mathcal{K}_q(\rho,\zeta,\xi,v)$. Suppose that $\zeta(1-\vartheta)^v s \geq 1, s = 4.567802$. Then $\mathcal{S}_l(z,f) \in \mathcal{K}_q(\rho,\zeta,\gamma,v)$, where

$$\gamma = \frac{\rho(2\xi-1)+\zeta(1-\vartheta)^v}{\rho+\zeta(1-\vartheta)^v}.$$

Proof For $f \in \mathcal{K}_q(\rho,\zeta,\xi,v)$, (5) gives

$$Re\left(\frac{\mathfrak{B}^q_{\zeta,\rho,v}f(z)^\rho}{\rho^q z^\rho}\right) = Re\left(1+\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^q z^{k-1}\right) > \xi. \tag{23}$$

Multiplying both sides of (23) by 2ρ, then adding $\zeta(1-\vartheta)^v$ to both sides, we obtain

$$Re\left(2\rho+\zeta(1-\vartheta)^v+2\rho\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^q z^{k-1}\right) > 2\xi\rho+\zeta(1-\vartheta)^v. \tag{24}$$

This implies

$$Re\left(\rho+\zeta(1-\vartheta)^v+2\rho\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^q z^{k-1}\right) > 2\xi\rho+\zeta(1-\vartheta)^v-\rho.$$

Equivalently,

$$Re\left(1+\frac{2\rho}{\rho+\zeta(1-\vartheta)^v}\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^q z^{k-1}\right) > \frac{2\xi\rho+\zeta(1-\vartheta)^v-\rho}{\rho+\zeta(1-\vartheta)^v}. \tag{25}$$

Now consider the partial sum $\mathcal{S}_l(z,f)$. Then

$$\frac{\mathfrak{B}^{q-1}_{\zeta,\rho,v}\mathcal{S}_l(z,f)^\rho}{\rho^{q-1}z^\rho} = 1+\sum_{k=2}^{l}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^{q-1}z^{k-1}.$$

Using the definition of convolution, we may write

$$\frac{\mathfrak{B}^{q-1}_{\zeta,\rho,v}\mathcal{S}_l(z,f)^\rho}{\rho^{q-1}z^\rho} = \left(1+\frac{2\rho}{\rho+\zeta(1-\vartheta)^v}\sum_{k=2}^{\infty}A_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^v}{\rho}\right)^q z^{k-1}\right)$$
$$*\left(1+\frac{\rho+\zeta(1-\vartheta)^v}{2\zeta(1-\vartheta)^v}\sum_{k=2}^{l}\frac{z^{k-1}}{\frac{\rho}{\zeta(1-\vartheta)^v}+(k-1)}\right). \tag{26}$$

By application of Lemma 2.3, it is observed that for $\zeta(1-\vartheta)^v \geq \frac{1}{s},\ s = 0.21892$, we have

$$Re\left(\sum_{k=2}^{l}\frac{z^{k-1}}{\frac{\rho}{\zeta(1-\vartheta)^v}+(k-1)}\right) > -\frac{\zeta(1-\vartheta)^v}{\rho+\zeta(1-\vartheta)^v}. \tag{27}$$

Hence,

$$Re\left(\frac{\rho+\zeta(1-\vartheta)^v}{\zeta(1-\vartheta)^v}\sum_{k=2}^{l}\frac{z^{k-1}}{\frac{\rho}{\zeta(1-\vartheta)^v}+(k-1)}\right) > -1. \tag{28}$$

Adding $+2$ to both sides of (28) yields

$$Re\left(2+\frac{\rho+\zeta(1-\vartheta)^{v}}{\zeta(1-\vartheta)^{v}}\sum_{k=2}^{l}\frac{z^{k-1}}{\frac{\rho}{\zeta(1-\vartheta)^{v}}+(k-1)}\right)>1. \quad (29)$$

Hence

$$Re\left(1+\frac{\rho+\zeta(1-\vartheta)^{v}}{2\zeta(1-\vartheta)^{v}}\sum_{k=2}^{l}\frac{z^{k-1}}{\frac{\rho}{\zeta(1-\vartheta)^{v}}+(k-1)}\right)>\frac{1}{2}$$

Finally, by Lemma 2.1 and the convolution representation in equation (26), we conclude that

$$Re\left(\frac{\mathfrak{B}_{\zeta,\rho,v}^{q-1}\mathcal{S}_l(z,f)^{\rho}}{\rho^{q-1}z^{\rho}}\right)>\gamma,$$

where

$$\gamma=\frac{\rho(2\xi-1)+\zeta(1-\vartheta)^{v}}{\rho+\zeta(1-\vartheta)^{v}}.$$

Hence,

$$\mathcal{S}_l(z,f)\in\mathcal{K}_q(\rho,\zeta,\gamma,v).$$

Corollary 3.3

Let $\mathcal{S}_l(z,f)$ be the l-th partial sum of a function of $f\in\mathcal{K}_q(\rho,\zeta,\xi,0)$ and suppose that $\zeta(1-\vartheta)^{v}s\geq 1, s=4.567802$. Then $\mathcal{S}_l(z,f)\in\mathcal{K}_q(\rho,\zeta,\gamma,0)$, where

$$\gamma=\frac{\rho(2\xi-1)+\zeta}{\rho+\zeta}.$$

Proof Setting $v=0$ in the Theorem 3.3 yields the required result.

Remark 3.3 If we set $v=0$ and $\rho=1$ in Corollary 1, we obtain the corresponding result for functions in $R(\xi,\zeta)$ given by (Al-Oboudi, 2004).

Theorem 3.4

Let $f,g\in\mathcal{K}_q(\rho,\zeta,\xi,v)$. Then $f*g\in\mathcal{K}_q(\rho,\zeta,\omega,v)$, where

$$\omega=\frac{(2\xi+1)(\rho+\zeta(1-\vartheta)^{v}+1)-3}{2(\rho+\zeta(1-\vartheta)^{v})}.$$

Proof Consider the sequence $\{u_k\}_{k=0}^{\infty}$ defined by

$$u_k=\begin{cases}1, & k=0,\\ \dfrac{(\rho+\zeta(1-\vartheta)^{v})\rho^{q}}{(\rho+\zeta(1-\vartheta)^{v}k)^{q}}, & k\geq 1,\end{cases}$$

where $q\in N$ and $0<\rho\leq 1$. Then $\{u_k\}_{k=0}^{\infty}$ is a convex null sequence, It follows that by Lemma 2.2, the function

$$\frac{\chi(z)^{\rho}}{z^{\rho}}=\frac{1}{2}+\sum_{k=2}^{l}\frac{(\rho+\zeta(1-\vartheta)^{v})\rho^{q}z^{k-1}}{(\rho+(k-1)\zeta(1-\vartheta)^{v})^{q}}, \quad (30)$$

is analytic in $\mathbb{E}$ and satisfies

$$Re\left(\frac{\chi(z)^{\rho}}{z^{\rho}}\right)=\mathrm{Re}\left(\frac{1}{2}+\sum_{k=2}^{l}\frac{(\rho+\zeta(1-\vartheta)^{v})\rho^{q}z^{k-1}}{(\rho+(k-1)\zeta(1-\vartheta)^{v})^{q}}\right)>0.$$

Equivalently,

$$\mathrm{Re}\left(1+\sum_{k=2}^{l}\frac{(\rho+\zeta(1-\vartheta)^{v})\rho^{q}z^{k-1}}{(\rho+(k-1)\zeta(1-\vartheta)^{v})^{q}}\right)>\frac{1}{2}. \quad (31)$$

Let $g\in\mathcal{K}_q(\rho,\zeta,\xi,v)$, where

$$g(z)^{\rho}=z^{\rho}+\sum_{k=2}^{\infty}\mathcal{B}_k(\rho)z^{\rho+k-1}.$$

Then

$$Re\left(\frac{\mathfrak{B}_{\zeta,\rho,v}^{q}g(z)^{\rho}}{\rho^{q}z^{\rho}}\right)=Re\left(1+\sum_{k=2}^{\infty}\mathcal{B}_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^{v}}{\rho}\right)^{q}z^{k-1}\right)>\xi. \quad (32)$$

Using the definition of convolution, we have

$$\frac{\mathfrak{B}_{\zeta,\rho,v}^{q}g(z)^{\rho}}{\rho^{q}z^{\rho}}*\frac{\chi(z)^{\rho}}{z^{\rho}}=\frac{\mathfrak{B}_{\zeta,\rho,v}^{q}g(z)^{\rho}*\chi(z)^{\rho}}{\rho^{q}z^{\rho}}$$

$$= \left(1+\sum_{k=2}^{\infty}\mathcal{B}_k(\rho)\left(\frac{\rho+(k-1)\zeta(1-\vartheta)^{v}}{\rho}\right)^{q} z^{k-1}\right) * \left(1+\sum_{k=2}^{l}\frac{(\rho+\zeta(1-\vartheta)^{v})\rho^{q} z^{k-1}}{(\rho+(k-1)\zeta(1-\vartheta)^{v})^{q}}\right)$$

$$= 1+(\rho+\zeta(1-\vartheta)^{v})\sum_{k=2}^{\infty}\mathcal{B}_k(\rho) z^{k-1}. \tag{33}$$

Hence,

$$Re\left(1+(\rho+\zeta(1-\vartheta)^{v})\sum_{k=2}^{\infty}\mathcal{B}_k(\rho) z^{k-1}\right) > \xi, \tag{34}$$

$$Re\left((\rho+\zeta(1-\vartheta)^{v})\sum_{k=2}^{\infty}\mathcal{B}_k(\rho) z^{k-1}\right) > \xi-1, \tag{35}$$

Therefore,

$$Re\left(\sum_{k=2}^{\infty}\mathcal{B}_k(\rho) z^{k-1}\right) > \frac{\xi-1}{(\rho+\zeta(1-\vartheta)^{v})}. \tag{36}$$

Adding $+1$ to both sides, gives

$$Re\left(\frac{g(z)^{\rho}}{z^{\rho}}\right) = Re\left(1+\sum_{k=2}^{\infty}\mathcal{B}_k(\rho) z^{k-1}\right) > \frac{\xi+\rho+\zeta(1-\vartheta)^{v}-1}{\rho+\zeta(1-\vartheta)^{v}}. \tag{37}$$

Subtracting $\frac{1}{2}$ from both sides, we obtain

$$Re\left(\frac{g(z)^{\rho}}{z^{\rho}}-\frac{1}{2}\right) > \frac{\xi+\rho+\zeta(1-\vartheta)^{v}-1}{\rho+\zeta(1-\vartheta)^{v}}-\frac{1}{2}. \tag{38}$$

Hence,

$$Re\left(\frac{g(z)^{\rho}}{z^{\rho}}-\frac{2\xi+\rho+\zeta(1-\vartheta)^{v}-2}{2(\rho+\zeta(1-\vartheta)^{v})}\right) > \frac{1}{2}. \tag{39}$$

Since $f \in \mathcal{K}_q(\rho,\zeta,\xi,v)$, an application of Lemma 2.1 yields

$$Re\left(\frac{\mathfrak{B}^{q}_{\zeta,\rho,v}f(z)^{\rho}}{\rho^{q} z^{\rho}} * \left(\frac{g(z)^{\rho}}{z^{\rho}}-\frac{2\xi+\rho+\zeta(1-\vartheta)^{v}-2}{2(\rho+\zeta(1-\vartheta)^{v})}\right)\right) > \xi, \tag{40}$$

Consequently,

$$Re\left(\frac{\mathfrak{B}^{q}_{\zeta,\rho,v}(f(z)^{\rho} * g(z)^{\rho})}{\rho^{q} z^{\rho}}\right) > \frac{(2\xi+1)(\rho+\zeta(1-\vartheta)^{v}+1)-3}{2(\rho+\zeta(1-\vartheta)^{v})}.$$

Therefore,

$$f * g \in \mathcal{K}_q(\rho,\zeta,\omega,v),$$

where

$$\omega = \frac{(2\xi+1)(\rho+\zeta(1-\vartheta)^{v}+1)-3}{2(\rho+\zeta(1-\vartheta)^{v})}.$$

This completes the proof.

Corollary 3.4 Let $f, g \in \mathcal{K}_q(\rho,\zeta,\xi,0)$. Then $f * g \in \mathcal{K}_q(\rho,\zeta,\gamma,0)$, where

$$\gamma = \frac{(2\xi+1)(\rho+\zeta+1)-3}{2(\rho+\zeta)}.$$

Proof Setting $v = 0$ in the Theorem 3.4, then the proof follows.

Remark 3.4 If we substitute $\zeta = 0$ and $\rho = 1$ in Corollary 3.4, we obtain the corresponding result for functions in $R(\xi)$ given by Ahuja (Ahuja and Jahangiri, 1998).

Remark 3.5 If we set $v = 0$ and $\rho = 1$ in Corollary 3.4, we obtain the corresponding result for functions in $R(\xi,\zeta)$ given by Al-Oboudi (Al-Oboudi, 2004).

DISCUSSION

Table 1 compares new and existing results. Theorem 2 reveals a nested hierarchy with respect to the parameter q. This suggests that the geometric conditions imposed by the generalized operator strengthen as the operator's order increases. Inclusion relations are crucial in geometric function theory, as they provide insight into the behavior and stability of subclasses under repeated applications of differential operators.

Theorem 3's partial sum result shows that $\mathcal{K}_q(\rho,\zeta,\omega,v)$ preserves key geometric properties under truncation. The derived lower bound for the real part indicates partial sums stay within a related subclass characterized by γ.

Theorem 4 reveals a quasi-convolution closure structure for the class. Using the Hadamard product and lemmas for functions with positive real part, the convolution of two functions in the class stays within a related subclass

Table 1: Comparison between the result of this study and existing result.

S/N	New Result	Existing Result(s)	Similarity (ies) and Difference(s)
1.	$\mathfrak{B}^q_{\zeta,\rho,v} f(z)^\rho = \rho^q z^\rho + \sum_{k=2}^{\infty} [\rho + \zeta(1-\vartheta)^v (k-1)]^q A_k(\rho) z^{\rho+k-1}$	Bello-Babalola Operator: $\mathfrak{B}^q_{\zeta,\rho} f(z)^\rho = \rho^q z^\rho + \sum_{k=2}^{\infty} [\rho + \zeta(k-1)]^q A_k(\rho) z^{\rho+k-1}$ Al-Oboudi Operator: $D^q_\zeta f(z) = z + \sum_{k=2}^{\infty} [1 + \zeta(k-1)]^q a_k z^k$ Sălăgean Differential Operator: $D^q f(z) = z + \sum_{k=2}^{\infty} k^q a_k z^k$	$\mathfrak{B}^0_{\zeta,1,v} f(z) = D^0_{\zeta,1,0} f(z) = D^q_\zeta f(z) = D^q f(z) = f(z)$ $\mathfrak{B}^q_{\zeta,\rho,0} f(z)^\rho = D^q_{\zeta,\rho} f(z)^\rho$ $\mathfrak{B}^q_{\zeta,1,0} f(z) = D^q_\zeta f(z)$ $\mathfrak{B}^q_{\zeta,1,0} f(z) = D^q f(z)$ They are linear operators

.

CONCLUSION

In this study, a generalized differential operator associated with a binomial series and a new subclass of analytic and univalent functions were introduced and investigated. The proposed operator and the class extend several well-known operators and their associated subclasses, respectively, in geometric function theory. These include the Sălăgean, Al-Oboudi, and Bello–Babalola operators as special cases. By employing convolution (Hadamard product) techniques together with classical results for Carathedory functions, several important properties of the newly defined subclasses were obtained. Specifically, convolution properties, inclusion relations, and partial sum results were established for functions belonging to the class. Furthermore, the results obtained generalize and unify several previously known results in the literature, thereby providing a unified approach for the study of subclasses of analytic and univalent functions defined via generalized differential operators. The examples presented give more insight into the applicability of the proposed operator. The generalized operator is expected to stimulate further investigations on coefficient inequalities, subordination properties, integral transforms, and other geometric characteristics of analytic and univalent function classes. Thus, the study contributes meaningfully to the continuing development of geometric function theory.

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